The Yoneda Lemma

Stephen Liu

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Abstract

Some notes on the Yoneda Lemma, starting with the notion of representable functors.

Covariant Representable Functors

We first define a *prototype* of a covariant representable functor out of a locally small category $\mathscr{A}$.

Definition. Let $\mathscr{A}$ be a locally small category, and fix $A \in \mathscr{A}$. Define the functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ by the following mapping on

1. **objects**: For $B \in \mathscr{A}$, define $H^A(B) = \mathscr{A}(A, B)$, the hom-set of arrows in $\mathscr{A}$ from A to B.
2. **morphisms**: For $g : B \to B'$, define $H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ by $p \mapsto g \circ p$ for all $p : A \to B$, sending morphisms from A to B to morphisms from A to B' by post-composition with g.

From this prototype of a representable functor, we can now define covariant representable functors:

Definition (Covariant Representable Functor)**.** Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A} \to \mathbf{Set}$ is representable if X is naturally isomorphic to H^A for some $A \in \mathscr{A}$. A representation of X is a choice of an object A along with a natural isomorphism from H^A to X.

Some examples of representable functors:

Example 1 (group G regarded as a one object category)**.** Regarding a group G as a one object category $\mathscr{G}$, with object $*$, $H^* = *(*, -) : \mathscr{G} \to \mathbf{Set}$ is a functor which maps

1. **objects**: There is only one object in $\mathscr{G}$, and $H^*(*)$ is the set of mappings from $*$ to $*$, otherwise known as the elements of the group G. So $H^*(*) \cong U(G)$ where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor that returns the underlying set of a group.

2. **morphisms**: Let $g \in G$, then $H^*(g)$ maps any element h of G to $g \circ h$, which, interpreted in the context of a group is just group mulitiplication on the left, i.e. gh.

Since $\mathscr{G}$ has only one object, there is only one representable functor on it (up to isomorphism), and the representable is the underlying set G acted on by left multiplication.

Example 2 ($H^1 : \mathbf{Set} \to \mathbf{Set}$)**.** Let 1 denote the set with just one element. The functor $1_{\mathbf{Set}}$ is represented by the functor $H^1 : \mathbf{Set} \to \mathbf{Set}$. To see this, let us first see what H^1 does and then show that it is naturally isomorphic to $1_{\mathbf{Set}}$.

Let $B \in \mathbf{Set}$, then $H^1(B) = \mathbf{Set}(1, B) = \{b : 1 \to B\}$ the set of functions from the singleton set to B that pick out an element of B. In particular, $H^1(B) \cong B$. Let $p : 1 \to B$ and $g : B \to B'$, then $H^1(g) = \mathbf{Set}(1, g) : \mathbf{Set}(1, B) \to \mathbf{Set}(1, B')$ sends functions p that pick out an element of B to functions $g \circ p$ that pick out an element of B'.

So we have the parallel functors $H^1, 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and we know what they do. Now we construct a natural isomorphism $t : H^1 \Rightarrow 1_{\mathbf{Set}}$. We already have that $H^1(B) \cong B$, so the components of t are the isomorphisms $t_B : H^1(B) \to B$ (and of course $t_B^{-1} : B \to H^1(B)$). Now we need to check that the naturality square below commutes:

$$\begin{tikzcd}
H^1(B) \arrow[r, "H^1(g)"] \arrow[d, "t_B"'] & H^1(B') \arrow[d, "t_{B'}"] \\
B \arrow[r, "1_{\mathbf{Set}}(g) = g"'] & B'
\end{tikzcd}$$

where $g : B \to B'$.

It is enough to consider an element $b : 1 \to B$ from $H^1(B)$. b picks out an element of B, which we will also identify by $b(1) = b$. Going down from $H^1(B)$, $t_B(b : 1 \to B)$ gives us this element b, which is then sent by g to some element of B', say b'. On the other hand, going right from $H^1(B)$, $H^1(g)(b : 1 \to B) = g \circ b$ which gives us the map $b' : 1 \to B'$ that picks out $g(b(1)) = b'$. Finally, $t_{B'}(b')$ is precisely this element b' of B'. So in fact the naturality square commutes.

Hence $1_{\mathbf{Set}}$ is represented by H^1.

Example 3 ($H^{\mathbf{1}}$: **Cat** $\to$ **Set**)**.** Similarly to the example above, ob : **Cat** $\to$ **Set** which sends a small category to its underlying set of objects is represented by $H^{\mathbf{1}}$ where $\mathbf{1}$ is the one-object category. This is because when $\mathscr{B}$ is a category, $H^{\mathbf{1}}(\mathscr{B}) = \mathbf{Cat}(\mathbf{1}, \mathscr{B})$ which picks out objects of $\mathscr{B}$ and is isomorphic to ob$\mathscr{B}$. The proof that these functors are naturally isomorphic is essentially the same as the one for $1_{\mathbf{Set}}$ and H^1 in the example above.

Adjoints and Representables

Now we establish the claim that any set valued functor with a left adjoint is representable. We first prove the following lemma.

Lemma 1. *Let $\mathscr{A}, \mathscr{B}$ be locally small categories with functors $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$ such that $F \dashv G$. Fix $A \in \mathscr{A}$, then the functor*

$$\mathscr{A}(A, G(-)) : \mathscr{B} \to \textit{\textbf{Set}}$$

*that is, the composition $\mathscr{B} \xrightarrow{G} \mathscr{A} \xrightarrow{H^A}$ **Set** is representable.*

Proof. Because $F \dashv G$, we have the isomorphism $t^{-1}_{A,B} : \mathscr{A}(A, G(B)) \to \mathscr{B}(F(A), B)$ for each $B \in \mathscr{B}$. Now $H^{F(A)}$ maps B to $\mathscr{B}(F(A), B)$, and it maps a morphism $g : B \to B'$ to $\mathscr{B}(F(A), g) : \mathscr{B}(F(A), B) \to \mathscr{B}(F(A), B')$. So we suspect that the isomorphisms $t^{-1}_{A,B}$ gives rise to the natural isomorphism $t^{-1} : \mathscr{A}(A, G(-)) \Rightarrow H^{F(A)}$. We show that this is in fact the case.

We need to check naturality. Let $g : B \to B'$. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "\mathscr{A}(A, Gg) = H^A(Gg)"] \arrow[d, "t^{-1}_{A,B}"'] & \mathscr{A}(A, G(B')) \arrow[d, "t^{-1}_{A,B'}"] \\
\mathscr{B}(F(A), B) \arrow[r, "H^{F(A)}(g)"'] & \mathscr{B}(F(A), B')
\end{tikzcd}$$

It suffices to consider a map $f : A \to G(B)$ from $\mathscr{A}(A, G(B))$ as it travels around the diagram. Going down from $\mathscr{A}(A, G(B))$, $t^{-1}_{A,B}(f : A \to G(B))$ is a map $t^{-1}_{A,B}(f) : F(A) \to B$, and then $H^{F(A)}(g)(t^{-1}_{A,B}) = g \circ t^{-1}_{A,B}$ which is a map from $F(A)$ to B'. On the other hand, going right from $\mathscr{A}(A, G(B))$, $H^A(Gg)(f : A \to G(B)) = Gg \circ f$ which is a map from A to $G(B')$ and then $t^{-1}_{A,B'}(Gg \circ f)$ is a map from $F(A)$ to B'. So the question is whether $t^{-1}_{A,B'}(Gg \circ f) = g \circ t^{-1}_{A,B}(f)$, but that is precisely what it means for t^{-1} to be natural with respect to B (See [Liu18a]), so in fact $\mathscr{A}(A, G(-)) \cong H^{F(A)}$ and is therefore representable. $\square$

Now we are ready to prove the main claim of this subsection:

Theorem 1. *Any set-valued functor that has a left adjoint is representable.*

Proof. Let $G : \mathscr{A} \to \mathbf{Set}$ be a functor with left adjoint F. Let 1 denote the set with a single element. For all $A \in \mathscr{A}$, $G(A)$ is a set and by example 2 above, $H^1(G(A)) = \mathbf{Set}(1, G(A)) \cong G(A)$ naturally in A where H^1 is our functor from $\mathbf{Set}$ to $\mathbf{Set}$ in example 2 above. So $G \cong \mathbf{Set}(1, G(-))$ and by the lemma above, this means G is representable. $\square$

Covariant Embedding

Proposition 1. *Let $f : A' \to A$, then f induces a natural transformation $H^f : H^A \Rightarrow H^{A'}$ with components $H^f_B : H^A(B) \to H^{A'}(B)$ so that a map $p : A \to B$ in $H^A(B)$ gets mapped via precomposition with f to $p \circ f : A' \to B$.*

Proof. We need to show that this map H^f is indeed a natural transformation. Let $g : B \to B'$. This means checking that the naturality square:

$$\begin{tikzcd}
H^A(B) \arrow[r, "H^A(g)"] \arrow[d, "H^f_B"'] & H^A(B') \arrow[d, "H^f_{B'}"] \\
H^{A'}(B) \arrow[r, "H^{A'}(g)"'] & H^{A'}(B')
\end{tikzcd}$$

commutes.

It is sufficient to consider a map $p : A \to B$ in $H^A(B)$ as it travels around the square. Going down, we have first $H^f_B(p) = p \circ f$, and then to the right we have $H^{A'}(g)(p \circ f) = g \circ (p \circ f)$. Going to the right we have $H^A(g)(p) = g \circ p$, and then going down we have $H^f_{B'}(g \circ p) = (g \circ p) \circ f$. However, since morphism composition is associative, these two things are equal and so the square indeed commutes. $\square$

With this natural transformation between covariant representables, we have the following definition:

Definition. The *covariant embedding* is the functor $H^\bullet : \mathscr{A}^{\text{op}} \to [\mathscr{A}, \mathbf{Set}]$ defined by the following mapping on:

1. **objects:** For object A in $\mathscr{A}$, $H^\bullet(A) = H^A$.
2. **morphisms:** For morphism $f : A' \to A$, $H^\bullet(f) = H^f$.

We can do many of the same things with the dual case.

Contravariant Representable Functors

Again, we start with a definition of the *prototypical* contravariant representable functor:

Definition. Let $\mathscr{A}$ be a locally small category with object A. Define the functor $H_A : \mathscr{A}(-, A) : \mathscr{A}^{\text{op}} \to \mathbf{Set}$ by the following mapping on

1. **objects:** For object B in $\mathscr{A}$, define $H_A(B) = \mathscr{A}(B, A)$.
2. **morphisms:** For morphism $g : B' \to B$, define $H_A(g) = \mathscr{A}(g, A) : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$ by $p : B \to A \mapsto p \circ g : B' \to B \to A$.

Similar to the covariant case, any functor $F : \mathscr{A}^{\text{op}} \to \mathbf{Set}$ that is naturally isomorphic to a *prototypical* contravariant representable functor is called contravariantly representable.

Example 4. The functor $\mathscr{P} : \mathbf{Set}^{\text{op}} \to \mathbf{Set}$ which maps sets to their powersets and maps functions $g : B' \to B$ to $\mathscr{P}(g) = g^{-1}U$ for all $U \in \mathscr{P}(B)$ where $g^{-1}U = \{x' \in B' | g(x') \in U\}$ is naturally isomorphic to $H_2 : \mathbf{Set}^{\text{op}} \to \mathbf{Set}$ where 2 denotes the set with two elements.

To show this, we first need to realize that a subset of a set B is just a boolean mapping from the set to 2 ($f(b) = 1$ means b is in that subset). So $H_2(B) = \mathbf{Set}(B, 2) \cong \mathscr{P}(B)$. Define $t_B : H_2(B) \to \mathscr{P}(B)$ with this isomorphism. We show that the t_B's are natural in B and therefore define the components of a natural transformation $t : H_2 \Rightarrow \mathscr{P}$. Let B, B' be sets, and let g be a map from B' to B. To do this, we show that the following naturality square commutes:

$$\begin{tikzcd}
H_2(B) \arrow[r, "H_2(g)"] \arrow[d, "t_B"'] & H_2(B') \arrow[d, "t_{B'}"] \\
\mathscr{P}(B) \arrow[r, "\mathscr{P}(g)"'] & \mathscr{P}(B')
\end{tikzcd}$$

Consider a map $u : B \to 2$ that picks out an associated subset $U \subseteq B$. $t_B(u)$ gives precisely this subset U, and $\mathscr{P}(g)(U)$ gives the subset U' of B' that maps along g into U. On the other hand, $H_2(g)(u)$ precomposes g with u to give the mapping $u \circ g : B' \to B \to 2$ which picks out the subset U' of B' that, when mapped along g gives back U. And $t'_B(u \circ g)$ gives this U'. So the diagram commutes.

Hence, $\mathscr{P}$ is represented by H_2.

Contravariant Embedding

Similar to the covariant case, the map $f : A \to A'$ induces a natural transformation $H_f : H_A \Rightarrow H_{A'}$ with components $H_f^B : H_A(B) \to H_{A'}(B)$ such that a map $p : B \to A \mapsto f \circ p : B \to A \to A'$.

This gives rise to the following contravariant embedding (which we will call the Yoneda Embedding)

Definition (Yoneda Embedding)**.** Let $\mathscr{A}$ be a locally small category. The Yoneda Embedding is the functor $H_\bullet : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ defined by the following mapping on:

1. **objects:** For object A in $\mathscr{A}$, $H_\bullet(A) = H_A$.
2. **morphisms:** For morphism $f : A \to A'$, $H_\bullet(f) = H_f$.

Proposition 2. *$H_\bullet$ is injective on isomorphism classes of objects.*

Proof. Let A, A' be objects in $\mathscr{A}$ such that $H_A \cong H_{A'}$. We need to show $A \cong A'$, that is, find maps $f : A \to A'$, $g : A' \to A$ that are mutually inverse. Since $H_A \cong H_{A'}$ that means there are maps (natural transformations) $\alpha :$

$H_A \Rightarrow H_{A'}$ and $\beta : H_{A'} \Rightarrow H_A$ such that $\alpha \circ \beta = 1_{H_{A'}}$ and $\beta \circ \alpha = 1_{H_A}$. We need to use α and β to construct the f and g we need.

First let's consider the components of α. Let B be an object in $\mathscr{A}$. Then α_B is a map from $H_A(B) \to H_{A'}(B)$. Letting $B = A$, we have $\alpha_A : H_A(A) \to H_{A'}(A)$. Now, we don't know what morphisms there are in $H_A(A)$, but we certainly know that 1_A exists. And applying α_A to 1_A, we get a map $\alpha_A(1_A) : A \to A'$.

Similarly, considering $\beta_{A'}$ applied to $(1_{A'})$, we get a map $\beta_{A'}(1_{A'}) : A' \to A$.

So now we have our maps between A and A'. We need to show that they are mutually inverse.

Naturality of α in B means that for every $p : B \to B'$ the following naturality square holds (note the contravariance):

$$\begin{tikzcd}
H_A(B) \arrow[d, "\alpha_B"'] & H_A(B') \arrow[l, "- \circ p"'] \arrow[d, "\alpha_{B'}"] \\
H_{A'}(B) & H_{A'}(B') \arrow[l, "- \circ p"]
\end{tikzcd}$$

In particular, if $x : B' \to A$ is a map $H_A(B')$, then $\alpha_B(x \circ p) = \alpha_{B'}(x) \circ p$.

Again, what we need is a specialization of this general phenomenon. Letting $x = 1_A$, which means making $B = A'$ and $B' = A$, our p becomes a map from $A' \to A$ and by naturality above we have $\alpha_{A'}(1_A \circ p) = \alpha_A(1_A) \circ p$. Now p is a map from $A' \to A$, so letting $p = \beta_{A'}(1_{A'})$ we have that $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = \alpha_{A'}(1_A \circ \beta_{A'}(1_{A'})) = \alpha_{A'}(\beta_{A'}(1_{A'}))$. But since $\alpha_{A'}$ and $\beta_{A'}$ are mutually inverse, this just means $\alpha_A(1_A) \circ \beta_{A'}(1_{A'}) = 1_{A'}$, which is precisely what we wanted.

Similarly, we can show using the naturality of β and specializing to $1_{A'}$ that $\beta_{A'}(1_{A'}) \circ \alpha_A(1_A) = 1_A$.

Thus $A \cong A'$. □

The Yoneda Lemma

Theorem 2 (The Yoneda Lemma)**.** *Let $\mathscr{A}$ be a locally small category. Then $[\mathscr{A}^{op}, \textbf{\textit{Set}}](H_A, X) \cong X(A)$ naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \textbf{\textit{Set}}]$.*

Proof. First let's give a quick outline of the proof:

1. Produce a mapping $t_{A,X} : [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \to XA$.
2. Produce a mapping $t_{A,X}^{-1} : XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$.
3. Show that t, t^{-1} are mutually inverse.
4. Show that t is natural in A.
5. Show that t is natural in X.

We're not going to do these steps in order, instead we'll do these steps the way I did it when I came up with the proof, so you'll see my motivation for each successive step.

Let's begin with step 1. Let $\alpha : H_A \Rightarrow X$ be a natural transformation. Looking at α's components, $\alpha_B : H_A(B) \to XB$ sends a map $g : B \to A$ to the element of XB, $\alpha_B(g)$. $t_{A,X}$ needs to map α to one specific element of XA. Consider the special case where $B = A$. Then we have the component $\alpha_A : H_A(A) \to XA$. Since the only map we are certain exists in $H_A(A)$ is the identity map 1_A, this suggests we need to define $t_{A,X}(\alpha) = \alpha_A(1_A)$.

To check that this mapping is the correct one, let's first check that $t_{A,X}$ is natural in X. Let $F : X \Rightarrow X'$ be a natural transformation. $t_{A,X}$ natural in X means that the following naturality square holds:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)} \arrow[r, "F \circ -"] \arrow[d, "t_{A,X}"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X')} \arrow[d, "t_{A,X'}"] \\
XA \arrow[r, "F_A"'] & X'A
\end{tikzcd}$$

Let $\alpha : H_A \Rightarrow X$ be a natural transformation. Then going down first and then to the right via F_A we have $F_A(t_{A,X}(\alpha)) = F_A(\alpha_A(1_A))$. On the other hand going right first via $F \circ -$ and then down we have $t_{A,X'}(F \circ \alpha) = (F \circ \alpha)_A(1_A)$. And by compositionality, $(F \circ \alpha)_A(1_A) = F_A \circ \alpha_A(1_A) = F_A(\alpha_A(1_A))$. So $t_{A,X}$ is indeed natural in X.

Now let us check that $t_{A,X}$ is natural in A. We won't succeed yet but this will give us a hint as to how we should define $t_{A,X}^{-1}$. Let f be a map from A to A'. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)} \arrow[d, "t_{A,X}"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_{A'}, X)} \arrow[l, "- \circ H_f"'] \arrow[d, "t_{A',X}"] \\
XA & XA' \arrow[l, "Xf"]
\end{tikzcd}$$

Let $\alpha' : H_{A'} \Rightarrow X$ be a natural transformation. Then going down first and then left via Xf we have $Xf(t_{A',X}(\alpha')) = Xf(\alpha'_{A'}(1_{A'}))$. On the other hand, going to the left first via $-\circ H_f$ and then down, we have $t_{A,X}(\alpha' \circ H_f) = (\alpha'_A \circ H_f^A)(1_A) = \alpha'_A(f)$ (Because $H_f^A(1_A) = f \circ 1_A) = f$. So we need to have the equality

$$Xf(\alpha'_{A'}(1_{A'})) = \alpha'_A(f)$$

Which we can't prove just yet. However, this does give us a hint as to how we should define $t_{A,X}^{-1}$. Notice that $\alpha'_{A'}(1'_A)$ is an element of XA' and that for every $g : B \to A' \in H_{A'}(B)$, $g \mapsto Xg(\alpha'_{A'}(1'_A))$ defines a mapping from $H_{A'}(B)$ to XB, which is beginning to look like the components of a natural transformation $H_{A'} \Rightarrow X$.

With that let us define our mapping $t_{A,X}^{-1} : XA \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$. Let x be an element of XA. We define $t_{A,X}^{-1}(x) = (X(f : B \to A)(x))_{B \in \mathscr{A}}$. So for each B, we have a family of morphisms $H_A(B) \to XB$ given by $f \mapsto Xf(x)$. Now we need to check that this family of morphisms does indeed form a natural transformation $H_A \Rightarrow X$.

Let $k : B \to B'$. We need to check that the following naturality square commutes:

$$\begin{tikzcd}
H_A(B) \arrow[d, "X(- : B \to A)(x)"'] & H_A(B') \arrow[l, "- \circ k"'] \arrow[d, "X(-' : B' \to A)(x)"] \\
XB & XB' \arrow[l, "Xk"]
\end{tikzcd}$$

Let $p : B' \to A$. Then going left and down we first have $p \circ k : B \to A$ which then becomes $X(p \circ k)(x)$. On the other hand, going down first and then left we have $Xk(Xp(x))$. These expressions are equal by the (co)functoriality of X. So $t^{-1}_{A,X}(x)$ does indeed yield a natural transformation.

Now we need to check that $t_{A,X}$ and $t^{-1}_{A,X}$ are in fact mutually inverse. Let $x \in XA$. Then $t_{A,X}(t^{-1}_{A,X}(x)) = t_{A,X}(X(-)(x)) = (X(-)(x))_A(1_A) = X1_A(x) = 1_{XA}(x) = x$. So $t_{A,X} \circ t^{-1}_{A,X}$ does indeed equal 1_{XA}. The other direction is a little bit harder:

Let α be a natural transformation from $H_A \Rightarrow X$. $t^{-1}_{A,X}(t_{A,X}(\alpha)) = t^{-1}_{A,X}(\alpha_A(1_A)) = X(-)(\alpha_A(1_A))$. Let $f : B \to A$. For $X(-)(\alpha_A(1_A)) = \alpha$, we need to have $Xf(\alpha_A(1_A)) = \alpha_B(f)$ for every f. The only tool we have at our disposal is naturality, either naturality of $X(-)(x)$ which we just established, or the naturality of α. When I first went about trying to prove this I thought we ought to use the naturality of $X(-)(x)$, however it turns out that the correct thing to do is use the naturality of α. (Thank you [Lei14]!) Let's write out the naturality square for α with f:

$$\begin{tikzcd}
H_A(B) \arrow[d, "\alpha_B"'] & H_A(A) \arrow[l, "- \circ f"'] \arrow[d, "\alpha_A"] \\
XB & XA \arrow[l, "Xf"]
\end{tikzcd}$$

Considering the image of 1_A as it moves around the square, we have $\alpha_B(1_A \circ f) = \alpha_B(f) = Xf(\alpha_A(1_A))$, which is exactly what we needed.

So t and t^{-1} are mutually inverse, and in fact this last step is also what we needed to prove that $t_{A,X}$ is natural in A, so we are done. □

Consequences of the Yoneda Lemma

One consequence of the Yoneda Lemma is that it justifies the name of the Yoneda Embedding, in that the Yoneda Embedding embeds $\mathscr{A}$ into a full subcategory of its presheaf category:

Proposition 3. *The Yoneda Embedding:* $H_\bullet : \mathscr{A} \to [\mathscr{A}^{op}, \boldsymbol{Set}]$ *is full and faithful.*

Proof. $H_\bullet$ being full and faithful is another way of saying the map $K : \mathscr{A}(A, A') \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_\bullet(A), H_\bullet(A'))$ is bijective for all $A, A' \in \mathscr{A}$. First recall that $H_\bullet(A) = H_A$, $H_\bullet(A') = H_{A'}$ and that $\mathscr{A}(A, A') = H_{A'}(A)$. So K is actually a map from $H_{A'}(A) \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, H_{A'})$. Remember that K comes from the Yoneda Embedding, which means K takes a morphism $f : A \to A'$ and maps it to $H_f : H_A \to H_{A'}$.

Well, by the Yoneda Lemma, we know there is a bijection $t^{-1}_{A,H_{A'}} : H_{A'}(A) \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, H_{A'})$. If K and $t^{-1}_{A,H_{A'}}$ were actually the same, then that would show that K is bijective. Let's consider what $t^{-1}_{A,H_{A'}}$ actually does. Let $f : A \to A'$. Then $f \in H_{A'}(A)$ and $t^{-1}_{A,H_{A'}}(f) = H_{A'}(g)(f) = f \circ g$ for all $g : B \to A$. So $t^{-1}_{A,H_{A'}}(f)$ is post-composition with f, which is exactly what H_f does, so K and $t^{-1}_{A,H_{A'}}$ are in fact the same. $\square$

Another consequence of the Yoneda Lemma is that isomorphic objects have isomorphic representables:

Proposition 4. *$H_A \cong H_{A'}$ if and only if $A \cong A'$.*

Proof. We have already shown that $H_A \cong H_{A'}$ implies $A \cong A'$ (See Proposition 2). Now suppose $A \cong A'$. Then since the embedding $\mathscr{A}(A, A') \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, H_{A'})$ is full and faithful (by Proposition 3), that means $H_\bullet(A) \cong H_\bullet(A')$, which means $H_A \cong H_{A'}$. $\square$

Let's look at some examples of this:

Example 5. Suppose we are in the category of groups, and that for two particular groups A, A', we know that for all B (though not necessarily naturally in B), $H_A(B) \cong H_{A'}(B)$. What can we say about A, A'? Let's substitute some groups for B and see what happens:

1. $B = 1$. Then we have that $\mathbf{Grp}(1, A) \cong \mathbf{Grp}(1, A')$. But the only group homomorphisms $\varphi : 1 \to A$ and $\varphi' : 1 \to A'$ are the ones sending 1 to the identity. So this simply says the set comprising the identity in A is isomorphic to the set comprising the identity in A', which is not very interesting.

2. $B = \mathbb{Z}$. Then since a group homomorphism $\varphi : \mathbb{Z} \to A$ that sends 1 to an element $a \in A$ picks out the cyclic subgroup of A generated by a, the set of group homomorphisms from $\mathbb{Z}$ to A picks out the underlying set of A (it acts like the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$). So $H_A(\mathbb{Z}) \cong H_{A'}(\mathbb{Z})$ means the underlying sets of A and A' are

isomorphic. However, this stops short of saying the group structures of A and A' are the same.

However, if we knew that this isomorphism of representables was natural in B, then we would be able to say that the groups were isomorphic.

Example 6. The category of sets is different, since a set A is isomorphic to the set of all functions from 1 to A, if we have $\mathbf{Set}(1, A) = H_A(1) \cong H_{A'}(1) = \mathbf{Set}(1, A')$, then we automatically know that $A \cong A'$.

Looking at tensor products now, it makes sense to talk about *the* tensor product of two vector spaces:

Example 7. In [Liu18b], we showed that the functor $\mathbf{Bilin}(U, V; -)$ is represented by the functor $\mathbf{Vect}(T, -) : \mathbf{Vect} \to \mathbf{Set}$ where T is a tensor product of U, V. So if we have $\mathbf{Vect}(T, -) \cong \mathbf{Vect}(T', -)$, that means $T \cong T'$, and the tensor product of U, V is unique up to isomorphism.

In a similar way, it makes sense to talk about *the* left adjoint of a certain functor, since they are unique up to isomorphism:

Example 8. Suppose $G : \mathscr{B} \to \mathscr{A}$ is a functor and that $F, F' : \mathscr{A} \to \mathscr{B}$ are both left adjoint to G. Then by the definition of adjoint functors, for each $A \in \mathscr{A}$, we have

$$\mathscr{B}(FA, B) \cong \mathscr{A}(A, GB) \cong \mathscr{B}(F'A, B)$$

naturally in B, which means $H^{FA} \cong H^{F'A}$. And by Proposition 4, we have $FA \cong F'A$, but since this is natural in A, this means $F \cong F'$.

References

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