Modules and Tensor Products

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Abstract

Some notes on modules and tensor products of modules.

Modules

The Basics

Definition (Modules over a ring)**.** Let R be a ring. A left R-module M is an abelian group $(M, +)$ with a map $R \times M \to M$ (also known as an action of R over M) such that for all $r, s \in R$, $m, n \in M$ we have:

1. $(r + s)m = rm + sm$
2. $r(m + n) = rm + rn$
3. $(rs)m = r(sm)$
4. $1m = m$ (If R contains 1).

We can define right R-modules analogously. Note that when R is a field, then a module over a field is precisely the same thing as a vector space over that field.

Definition (Submodules)**.** Let M be a R-module. A submodule of M is a subgroup N of M that is closed under the ring action, that is, $rn \in N$ for all $r \in R$, $n \in N$ (in the case of left R-modules).

One important example of a module are the $\mathbb{Z}$-modules:

Example 1 ($\mathbb{Z}$-Modules)**.** Consider the ring $\mathbb{Z}$ and any abelian group A. Then we can make A into a $\mathbb{Z}$-module by defining the action $\mathbb{Z} \times A \to A$ by

$$na = \begin{cases} a + a + \cdots + a \quad \text{(n times)} & n > 0 \\ 0 & n = 0 \\ -a - a - \cdots - a \quad \text{(n times)} & n < 0 \end{cases}$$

So any abelian group A is a $\mathbb{Z}$-module. Conversely, it turns out that every $\mathbb{Z}$-module is an abelian group.

Now we define the notion of module homomorphisms.

Definition (Module Homomorphisms)**.** Let M and N be R-modules. A function $\varphi : M \to N$ is a module homomorphism if for all $r \in R$ and $x, y \in M$ we have

1. $\varphi(x + y) = \varphi(x) + \varphi(y)$
2. $\varphi(rx) = r\varphi(x)$

Now our goal is to arrive at a definition of tensor products of modules, which will involve free Z-modules, so let's first go over the definition of a free module and an important universal property of free modules.

Freely Generated Modules

Definition (Free Modules)**.** An R-module F is free on a subset $A \subseteq F$ if for every nonzero $x \in F$, there are unique nonzero elements $r_1, r_2, \ldots, r_n \in R$ and unique $a_1, a_2, \ldots, a_n \in A$ such that $x = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some positive integer n. We call A a basis for F and that A is the set of free generators of F.

Notice that when R is a field, then A is the set of basis vectors (will also need linear independence) for the vector space F over the field R.

We now talk about an important universal property of free modules, which is a precursor to the defining universal property of tensor products.

Proposition 1 (Universal Property of Free Modules)**.** *For any set A there is a free R-module $F(A)$ on A such that if M is any R-module and $\varphi : A \to M$ is any map of sets, then we have the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, dashed, "\exists!\Phi"] \\
& M
\end{tikzcd}$$

where Φ is an R-module homomorphism.

Proof. By convention, if $A = \emptyset$ we define $F(A) = \{0\}$. In that case, φ is the unique map of sets $\emptyset \to M$, $F(A)$ is also the empty set and ι is the identity map, which means $\Phi = \varphi$. Otherwise, if A is nonempty, then let $F(A)$ be the collection of all set functions $f : A \to R$ such that $f(a) = 0$ for all but finitely many $a \in A$. We can make $F(A)$ into an R-module by pointwise addition of functions and pointwise multiplication of ring elements times a function, so we have for all $f, g \in F(A)$ and $r \in R$:

$$(f + g)(a) = f(a) + g(a)$$
$$(rf)(a) = r(f(a))$$

for all $x \in A$.

Let's just check to make sure this indeed gives us an R-module. Let $r, s \in R$, $f, g \in F(A)$. For each $a \in A$:

1. $(r+s)f$ gives $(r+s)(f(a))$ which equals $r(f(a))+s(f(a))$ (Since $f(a)$ is an element in R) which finally gives $rf + sf$. So $(r+s)f = rf+sf$.

2. $r(f + g)$ gives $r(f(a) + g(a))$ and since $f(a), g(a)$ are elements in R, this gives $r(f(a)) + r(g(a)) = rf + rg$.

3. $(rs)f = (rs)(f(a)) = r(s(f(a))) = r(sf)$.

So $F(A)$ is indeed an R-module. Now we need to show $F(A)$ is freely generated by A. We define the map $\iota : A \to F(A)$ by $a \mapsto f_a$ where

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$$

Since ι is injective (Let $a, b \in A$ such that $f_a = f_b$, then f_a and f_b both take the value 1 at the same point x which is both equal to a and equal to b, so $a = b$), we see that ι can be seen as an embedding of A in $F(A)$. This allows us to view $F(A)$ as all finite R-linear combinations of elements of A in the following way:

Let $f : A \to R$ be a nonzero element of $F(A)$. Then by definition of $F(A)$, f takes on a nonzero value (in R) for finitely many points in A, say $a_1, a_2, \ldots, a_n$. So at each a_i, f takes on a nonzero value, say r_i. That means we can uniquely write f as the R-linear combination $r_1 f_{a_i} + r_2 f_{a_2} + \cdots + r_n f_{a_n}$. Hence, $F(A)$ is indeed freely generated by A.

Now given the map on sets $\varphi : A \to M$, we define $\Phi : F(A) \to M$ by $\sum_{i=1}^n r_i f_{a_i} \mapsto \sum_{i=1}^n r_i \varphi(a_i)$. Let's verify that Φ is indeed a well-defined R-module homomorphism. Let $r \in R$ and $f, g \in F(A)$. We have just established that f can be written uniquely as $\sum_{i=1}^n r_i f_{a_i}$ and likewise g can be written uniquely as $\sum_{k=1}^m s_k g_{b_k}$. Then:

1. *well-defined:* Since elements $f \in F(A)$ are written uniquely as formal R-linear sums of f_a's and φ is well-defined, Φ is well-defined.

2. $\Phi(rf) = \Phi(r \sum_{i=1}^n r_i f_{a_i}) = r \sum_{i=1}^n r_i \varphi(a_i) = r\Phi(f)$

3. $\Phi(f+g) = \Phi(\sum_{i=1}^n r_i f_{a_i} + \sum_{k=1}^m s_k g_{b_k}) = \Phi(r_1 f_{a_1} + r_2 f_{a_2} + \cdots + r_n f_{a_n} + s_1 g_{b_1} + s_2 g_{b_2} + \ldots s_m g_{b_m}) = r_1 \varphi(a_1) + r_2 \varphi(a_2) + \cdots + r_n \varphi(a_n) + s_1 \varphi(b_1) + s_2 \varphi(b_2) + \cdots + s_m \varphi(b_m) = \sum_{i=1}^n r_i \varphi(a_i) + \sum_{k=1}^m s_k \varphi(b_k) = \Phi(f) + \Phi(g)$.

Hence, Φ is a well-defined R-module homomorphism and by definition, Φ restricted to $A \subseteq F(A)$ is φ. Finally, since $F(A)$ is generated by A, which means the elements of $F(A)$ are uniquely written as formal R-linear sums of elements of A, once we know the values of φ on A, φ's values on elements of $F(A)$ are uniquely determined. So Φ is the unique extension of φ to all of $F(A)$. $\square$

Tensor Products of Modules

Basic Definition

We now have the algebraic framework we need to define the tensor product of modules:

Definition (Tensor Product of Modules)**.** Let R be a ring with right R-module M and left R-module N. Then the free $\mathbb{Z}$-module on the set $M \times N$, which we will write $\mathbb{Z}(M \times N)$ is the set of formal $\mathbb{Z}$-linear sums of elements $(m, n) \in M \times N$. Since this is a free $\mathbb{Z}$-module, it is an abelian group. Quotienting out the subgroup H generated by elements of the form

$$(m,(n_1+n_2)) - (m,n_1) - (m,n_2)$$
$$((m_1+m_2),n) - (m_1,n) - (m_2,n)$$
$$(mr,n) - (m,rn)$$

produces the abelian quotient group $\mathbb{Z}(M \times N)/H$ which we call the tensor product of M and N over R, written $M \bigotimes_R N$. We write cosets (m,n) in this abelian group as $m \otimes n$ and call them simple tensors in the tensor product. Elements of $M \bigotimes_R N$ are formal $\mathbb{Z}$-linear sums of simple tensors.

Note that quotienting out by that particular subgroup basically enforces the following relations (which we write with tensor notation now):

$$m \otimes (n_1+n_2) = m \otimes n_1 + m \otimes n_2$$
$$(m_1+m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$
$$mr \otimes n = m \otimes rn$$

Using these relations we can look at the following examples:

Example 2. For any tensor product $M \bigotimes_R N$ we have $m \otimes 0 = m \otimes (0+0) = m \otimes 0 + m \otimes 0$, so $m \otimes 0 = 0$. Similarly, $0 \otimes n = (0+0) \otimes n = 0 \otimes n + 0 \otimes n$, so $0 \otimes n = 0$.

Example 3. $\mathbb{Z}/n \bigotimes_R \mathbb{Z}/m = 0$ whenever n, m are relatively prime. This is because since n, m are relatively prime, for any $a \in \mathbb{Z}/n$, $ma = a$, so for any $a \in \mathbb{Z}/n, b \in \mathbb{Z}/m$, $a \otimes b = ma \otimes b = a \otimes mb = a \otimes 0 = 0$.

More examples to come...

Universal Property of Tensor Products

There is a canonical map $\iota : M \times N \to M \bigotimes_R N$ defined by $(m,n) \mapsto m \otimes n$.

Definition (R-balanced map)**.** Let M be a right R-module, N be a left R-module and L be an abelian group. Then a map $\varphi : M \times N \to L$ is R-balanced if it is linear in each variable and additionally, $\varphi(mr,n) = \varphi(m,rn)$ for all $m \in M, n \in N, r \in R$.

So the canonical map ι is R-balanced.

We now have the analogous universal property of the tensor product:

Proposition 2 (Universal Property of the Tensor Product)**.** *Let M be a right R-module, N be a left R-module and L be any abelian group. Then* *there is a bijection between R-balanced maps $\varphi : M \times N \to L$ and group homomorphisms $\Phi : M \bigotimes_R N \to L$ that satisfies the commutative triangle:*

$$\begin{tikzcd}
M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \bigotimes_R N \arrow[d, "\Phi"] \\
& L
\end{tikzcd}$$

Proof. In the first direction, let Φ be a group homomorphism from $M \bigotimes_R N \to L$. Then defining $\varphi = \Phi \circ \iota$, we have a map from $M \times N \to L$. We need to check that φ is in fact R-balanced. Let $m_1, m_2 \in M$ and $n \in N$. Then $\varphi(m_1 + m_2, n) = \Phi(\iota(m_1 + m_2, n)) = \Phi((m_1 + m_2) \otimes n) = \Phi(m_1 \otimes n + m_2 \otimes n)$. Since Φ is a group homomorphism, we have $\Phi(m_1 \otimes n + m_2 \otimes n) = \Phi(m_1 \otimes n) + \Phi(m_2 \otimes n) = \Phi(\iota(m_1, n)) + \Phi(\iota(m_2, n)) = \varphi(m_1, n) + \varphi(m_2, n)$. So φ is linear in M. Similarly we can show that φ is linear in N. Let $r \in R$, then $\varphi(mr, n) = \Phi(\iota(mr, n)) = \Phi(mr \otimes n) = \Phi(m \otimes rn) = \Phi(\iota(m, rn)) = \varphi(m, rn)$. So φ is R-balanced in $M \times N$.

In the other direction, using Proposition 1 the R-balanced map φ defines a $\mathbb{Z}$-module homomorphism φ' from the free $\mathbb{Z}$ module $\mathbb{Z}(M \times N)$ to L such that $\varphi'(m, n) = \varphi(m, n)$. Since φ is R-balanced, φ' maps elements of the form that generated the subgroup H in our definition of the tensor product to 0, so the kernel of φ' contains H. Hence, φ' induces a group homomorphism Φ on the quotient group $M \bigotimes_R N$ to L by $\Phi(m \otimes n) = \varphi'(m, n) = \varphi(m, n)$. Since the elements $m \otimes n$ generate $M \bigotimes_R N$, Φ is uniquely determined by this equation. $\square$

We now have to consider the module structure of the tensor product. Notice that if R is a commutative ring, then since $rm = mr$, $M \bigotimes_R N$ is a left R-module given by:

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn)$$

In this case, Proposition 2 gives a bijection between R-bilinear maps $M \times N \to L$ (no longer needed to be R-balanced because of this commutativity relation above) and R-module homomorphisms $M \bigotimes_R N \to L$.

We have the following fact about the tensor product:

Proposition 3 (Tensor Product is Associative)**.** *$X \bigotimes (Y \bigotimes Z) \cong (X \bigotimes Y) \bigotimes Z$.*

We actually have two proofs of this fact. The first uses only the universal property of tensor products, while the second uses an application of the Yoneda Lemma (see [Liu18]).

Proof using Universal Property. For each $x \in X$, we define a map

$$\begin{gathered} \phi : Y \times Z \to (X \bigotimes Y) \bigotimes Z \\ (y, z) \mapsto (x \otimes y) \otimes z \end{gathered}$$

This map is bilinear because $\phi(y_1 + y_2, z) = (x \otimes (y_1 + y_2)) \otimes z = ((x \otimes y_1) + (x \otimes y_2)) \otimes z = (x \otimes y_1) \otimes z + (x \otimes y_2) \otimes z$ and similarly for the z-coordinate.

Since ϕ is bilinear, by the universal property it induces a linear map

$$\begin{gathered} \Phi_x : Y \bigotimes Z \to (X \bigotimes Y) \bigotimes Z \\ y \otimes z \mapsto (x \otimes y) \otimes z \end{gathered}$$

Now we define a map

$$\begin{gathered} \delta : X \times (Y \otimes Z) \to (X \bigotimes Y) \bigotimes Z \\ (x, \sum_{i=1}^{n} y_i \otimes z_i) \mapsto \Phi_x(\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i \end{gathered}$$

Again, this map is bilinear (because of linearity of Φ_x and properties of the tensor product), so by the universal property it induces a linear map

$$\begin{aligned} \Delta : X \bigotimes (Y \otimes Z) &\to (X \bigotimes Y) \bigotimes Z \\ x \otimes (\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} x \otimes (y_i \otimes z_i) &\mapsto \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i \end{aligned}$$

We construct the inverse by fixing $z \in Z$ and proceeding in a similar way to get

$$\begin{gathered} \Gamma : (X \bigotimes Y) \otimes Z \to X \bigotimes (Y \bigotimes Z) \\ \sum_{i=1}^{n} (x_i \otimes y_i) \otimes z \mapsto \sum_{i=1}^{n} x_i \otimes (y_i \otimes z) \end{gathered}$$

Notice that in Δ there is only one x and many z and in Γ there is only one z and many x. One might think that this is a problem but we check that Δ, Γ are indeed mutually inverse:

$$\begin{aligned}
\Gamma(\Delta(\sum_{i=1}^{n} x \otimes (y_i \otimes z_i))) &= \Gamma(\sum_{i=1}^{n} (x \otimes y_i) \otimes z_i) \\
&= \sum_{i=1}^{n} \Gamma((x \otimes y_i) \otimes z_i) \quad \text{By linearity of } \Gamma \\
&= \sum_{i=1}^{n} x \otimes (y_i \otimes z_i)
\end{aligned}$$

and

$$\begin{aligned}
\Delta(\Gamma(\sum_{i=1}^{n} (x_i \otimes y_i) \otimes z)) &= \Delta(\sum_{i=1}^{n} x_i \otimes (y_i \otimes z)) \\
&= \sum_{i=1}^{n} \Delta(x_i \otimes (y_i \otimes z)) \quad \text{By linearity of } \Delta \\
&= \sum_{i=1}^{n} (x_i \otimes y_i) \otimes z
\end{aligned}$$

□

Proof using Yoneda Lemma. By the Yoneda Lemma, if two representable functors $H^A, H^{A'}$ are isomorphic, then that means $A \cong A'$. We first show that the functor

$$\begin{aligned}
&\mathbf{Bilin}(U, V; -) : \mathbf{Vect}_k \to \mathbf{Set} \\
&\qquad \text{Vector space } W \mapsto \text{set of bilinear maps } U \times V \to W
\end{aligned}$$

is representable by showing that it is naturally isomorphic to the functor $H^{U \bigotimes V} = \mathbf{Vect}(U \bigotimes V, -)$

We define the mapping $t_W : \mathbf{Bilin}(U, V; W) \to \mathbf{Vect}(U \bigotimes V, W)$ by $t_W(\varphi) = \Phi$ where Φ is the linear map out of the tensor product uniquely determined by φ given by the universal property of tensor products. We define $t_W^{-1} : \mathbf{Vect}(U \bigotimes V, W) \to \mathbf{Bilin}(U, V; W)$ by $t_W^{-1}(\Phi) = \varphi$. Clearly t_W, t_W^{-1} are inverse to each other. Let $f : W \to W'$. We need to show that the following naturality square holds:

$$
\begin{tikzcd}
\mathbf{Bilin}(U,V;W) \arrow[r, "f \circ -"] \arrow[d, "t_W"'] & \mathbf{Bilin}(U,V;W') \arrow[d, "t_{W'}"] \\
\mathbf{Vect}(U \bigotimes V, W) \arrow[r, "\mathbf{Vect}(U \bigotimes V, f) = f \circ -"'] & \mathbf{Vect}(U \bigotimes V, W')
\end{tikzcd}
$$

This clearly holds because of the following universal property:

$$
\begin{tikzcd}
U \times V \arrow[rr, "\iota"] \arrow[drr, "\varphi"] \arrow[ddrr, "f \circ \varphi"'] & & U \bigotimes V \arrow[d, "\Phi"] \arrow[dd, bend left, "f \circ \Phi"] \\
& & W \arrow[d, "f"] \\
& & W'
\end{tikzcd}
$$

So we have the following chain of isomorphisms:

$$
\begin{aligned}
\mathbf{Vect}(X \bigotimes (Y \bigotimes Z), -) &\cong \mathbf{Bilin}(X, Y \bigotimes Z; -) \\
&\cong \mathbf{3\text{-}lin}(X, Y, Z; -) \\
&\cong \mathbf{Bilin}(X \bigotimes Y, Z; -) \\
&\cong \mathbf{Vect}((X \bigotimes Y) \bigotimes Z, -)
\end{aligned}
$$

which will give us $X \bigotimes (Y \bigotimes Z) \cong (X \bigotimes Y) \bigotimes Z$. The only line in here we need to prove is the isomorphism $\mathbf{Bilin}(X, Y \bigotimes Z; -) \cong \mathbf{3\text{-}lin}(X, Y, Z; -)$. So we need to establish a bijection between bilinear maps $X \times (Y \bigotimes Z) \to W$ with 3-linear maps $X \times Y \times Z \to W$. Firstly, given a bilinear map $\varphi : X \times (Y \bigotimes Z) \to W$, we can define a 3-linear map $\tilde{\varphi} : X \times Y \times Z \to W$ by

$(x,y,z) \mapsto \varphi(x, \iota(y,z))$ where ι is the canonical map from $Y \times Z \to Y \bigotimes Z$. $\tilde{\varphi}$ is clearly 3-linear because φ was linear in x and separately linear in $\iota(y,z)$, while ι is bilinear.

Now going the other way suppose we had a 3-linear map $\varphi : X \times Y \times Z \to W$. How do we get a bilinear map out of $X \times (Y \bigotimes Z)$? Fixing $x \in X$, we have the bilinear map out of $\varphi_x : Y \times Z \to W$ by $(y,z) \mapsto \varphi(x,y,z)$. By the universal property of the tensor product, φ_x gives us a linear map $\delta_x : Y \bigotimes Z \to W$. We now define the bilinear map $\hat{\varphi} : X \times (Y \bigotimes Z) \to W$ by $(x, \sum\limits_{i=1}^{n} y_i \otimes z_i) \mapsto \delta_x(\sum\limits_{i=1}^{n} y_i \otimes z_i)$. Notice that this is a very similar construction in our first proof using the universal property. Since δ_x is linear, $\hat{\varphi}$ is clearly linear in the second argument. It remains to check that $\hat{\varphi}$ is linear in x:

$$\begin{aligned}
\hat{\varphi}(x_1 + x_2, \sum_{i=1}^{n} y_i \otimes z_i) &= \delta_{x_1+x_2}(\sum_{i=1}^{n} y_i \otimes z_i) \\
&= \sum_{i=1}^{n} \delta_{x_1+x_2}(y_i \otimes z_i) \\
&= \sum_{i=1}^{n} \delta_{x_1+x_2}(\iota(y_i, z_i)) \\
&= \sum_{i=1}^{n} \varphi_{x_1+x_2}(y_i, z_i) \quad \text{By definition of } \delta_x \\
&= \sum_{i=1}^{n} \varphi(x_1 + x_2, y_i, z_i) \quad \text{By definition of } \varphi_x \\
&= \sum_{i=1}^{n} \varphi(x_1, y_i, z_i) + \varphi(x_2, y_i, z_i) \quad \text{By trilinearity of } \varphi
\end{aligned}$$

Finally, it turns out that $\hat{\tilde{\varphi}} = \varphi$ and $\tilde{\hat{\varphi}} = \varphi$, so these are mutually inverse to each other. $\square$

References

[Lei14] Tom Leinster. *Basic Category Theory.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. The Yoneda Lemma, 2018. https://ssyl55.github.io/files/yoneda.pdf.