# Modules and Tensor Products

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#### Abstract

Some notes on modules and tensor products of modules.

# Modules

The Basics

**Definition** (Modules over a ring). Let R be a ring. A left R-module M is an abelian group (M, +) with a map  $R \times M \to M$  (also known as an action of R over M) such that for all  $r, s \in R, m, n \in M$  we have:

- 1. (r+s)m = rm + sm
- 2. r(m+n) = rm + rn
- 3. (rs)m = r(sm)
- 4. 1m = m (If R contains 1).

We can define right R-modules analogously. Note that when R is a field, then a module over a field is precisely the same thing as a vector space over that field.

**Definition** (Submodules). Let M be a R-module. A submodule of M is a subgroup N of M that is closed under the ring action, that is,  $rn \in N$  for all  $r \in R$ ,  $n \in N$  (in the case of left R-modules).

One important example of a module are the  $\mathbb{Z}$ -modules:

**Example 1** ( $\mathbb{Z}$ -Modules). Consider the ring  $\mathbb{Z}$  and any abelian group A. Then we can make A into a  $\mathbb{Z}$ -module by defining the action  $\mathbb{Z} \times A \to A$  by

$$na = \begin{cases} a + a + \dots + a & (n \text{ times}) & n > 0 \\ 0 & n = 0 \\ -a - a - \dots - a & (n \text{ times}) & n < 0 \end{cases}$$

So any abelian group A is a  $\mathbb{Z}$ -module. Conversely, it turns out that every  $\mathbb{Z}$ -module is an abelian group.

Now we define the notion of module homomorphisms.

**Definition** (Module Homomorphisms). Let M and N be R-modules. A function  $\varphi : M \to N$  is a module homomorphism if for all  $r \in R$  and  $x, y \in M$  we have

- 1.  $\varphi(x+y) = \varphi(x) + \varphi(y)$
- 2.  $\varphi(rx) = r\varphi(x)$

Now our goal is to arrive at a definition of tensor products of modules, which will involve free Z-modules, so let's first go over the definition of a free module and an important universal property of free modules.

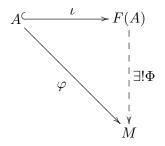
#### **Freely Generated Modules**

**Definition** (Free Modules). An *R*-module *F* is free on a subset  $A \subseteq F$  if for every nonzero  $x \in F$ , there are unique nonzero elements  $r_1, r_2, \ldots, r_n \in R$ and unique  $a_1, a_2, \ldots, a_n \in A$  such that  $x = r_1a_1 + r_2a_2 + \cdots + r_na_n$  for some positive integer *n*. We call *A* a basis for *F* and that *A* is the set of free generators of *F*.

Notice that when R is a field, then A is the set of basis vectors (will also need linear independence) for the vector space F over the field R.

We now talk about an important universal property of free modules, which is a precursor to the defining universal property of tensor products.

**Proposition 1** (Universal Property of Free Modules). For any set A there is a free R-module F(A) on A such that if M is any R-module and  $\varphi : A \to M$ is any map of sets, then we have the following commutative diagram:



where  $\Phi$  is an *R*-module homomorphism.

*Proof.* By convention, if  $A = \emptyset$  we define  $F(A) = \{0\}$ . In that case,  $\varphi$  is the unique map of sets  $\emptyset \to M$ , F(A) is also the empty set and  $\iota$  is the identity map, which means  $\Phi = \varphi$ . Otherwise, if A is nonempty, then let F(A) be the collection of all set functions  $f : A \to R$  such that f(a) = 0 for all but finitely many  $a \in A$ . We can make F(A) into an R-module by pointwise addition of functions and pointwise multiplication of ring elements times a function, so we have for all  $f, g \in F(A)$  and  $r \in R$ :

$$(f+g)(a) = f(a) + g(a)$$
$$(rf)(a) = r(f(a))$$

for all  $x \in A$ .

Let's just check to make sure this indeed gives us an *R*-module. Let  $r, s \in R, f, g \in F(A)$ . For each  $a \in A$ :

- 1. (r+s)f gives (r+s)(f(a)) which equals r(f(a)) + s(f(a)) (Since f(a) is an element in R) which finally gives rf + sf. So (r+s)f = rf + sf.
- 2. r(f+g) gives r(f(a) + g(a)) and since f(a), g(a) are elements in R, this gives r(f(a)) + r(g(a)) = rf + rg.

3. 
$$(rs)f = (rs)(f(a)) = r(s(f(a))) = r(sf)$$
.

So F(A) is indeed an *R*-module. Now we need to show F(A) is freely generated by *A*. We define the map  $\iota : A \to F(A)$  by  $a \mapsto f_a$  where

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$$

Since  $\iota$  is injective (Let  $a, b \in A$  such that  $f_a = f_b$ , then  $f_a$  and  $f_b$  both take the value 1 at the same point x which is both equal to a and equal to b, so a = b), we see that  $\iota$  can be seen as an embedding of A in F(A). This

allows us to view F(A) as all finite *R*-linear combinations of elements of *A* in the following way:

Let  $f: A \to R$  be a nonzero element of F(A). Then by definition of F(A), f takes on a nonzero value (in R) for finitely many points in A, say  $a_1, a_2, \ldots, a_n$ . So at each  $a_i$ , f takes on a nonzero value, say  $r_i$ . That means we can uniquely write f as the R-linear combination  $r_1 f_{a_i} + r_2 f_{a_2} + \cdots + r_n f_{a_n}$ . Hence, F(A) is indeed freely generated by A.

Now given the map on sets  $\varphi : A \to M$ , we define  $\Phi : F(A) \to M$ by  $\sum_{i=1}^{n} r_i f_{a_i} \mapsto \sum_{i=1}^{n} r_i \varphi(a_i)$ . Let's verify that  $\Phi$  is indeed a well-defined *R*-module homomorphism. Let  $r \in R$  and  $f, g \in F(A)$ . We have just established that f can be written uniquely as  $\sum_{i=1}^{n} r_i f_{a_i}$  and likewise g can be written uniquely as  $\sum_{k=1}^{m} s_k g_{b_k}$ . Then:

- 1. well-defined: Since elements  $f \in F(A)$  are written uniquely as formal *R*-linear sums of  $f_a$ 's and  $\varphi$  is well-defined,  $\Phi$  is well-defined.
- 2.  $\Phi(rf) = \Phi(r\sum_{i=1}^{n} r_i f_{a_i}) = r\sum_{i=1}^{n} r_i \varphi(a_i) = r\Phi(f)$
- 3.  $\Phi(f+g) = \Phi(\sum_{i=1}^{n} r_i f_{a_i} + \sum_{k=1}^{m} s_k g_{b_k}) = \Phi(r_1 f_{a_1} + r_2 f_{a_2} + \dots + r_n f_{a_n} + s_1 g_{b_1} + s_2 g_{b_2} + \dots + s_m g_{b_m}) = r_1 \varphi(a_1) + r_2 \varphi(a_2) + \dots + r_n \varphi(a_n) + s_1 \varphi(b_1) + s_2 \varphi(b_2) + \dots + s_m \varphi(b_m) = \sum_{i=1}^{n} r_i \varphi(a_i) + \sum_{k=1}^{m} s_k \varphi(b_k) = \Phi(f) + \Phi(g).$

Hence,  $\Phi$  is a well-defined *R*-module homomorphism and by definition,  $\Phi$  restricted to  $A \subseteq F(A)$  is  $\varphi$ . Finally, since F(A) is generated by *A*, which means the elements of F(A) are uniquely written as formal *R*-linear sums of elements of *A*, once we know the values of  $\varphi$  on *A*,  $\varphi$ 's values on elements of F(A) are uniquely determined. So  $\Phi$  is the unique extension of  $\varphi$  to all of F(A).

## **Tensor Products of Modules**

#### **Basic Definition**

We now have the algebraic framework we need to define the tensor product of modules:

**Definition** (Tensor Product of Modules). Let R be a ring with right R-module M and left R-module N. Then the free  $\mathbb{Z}$ -module on the set  $M \times N$ , which we will write  $\mathbb{Z}(M \times N)$  is the set of formal  $\mathbb{Z}$ -linear sums of elements  $(m, n) \in M \times N$ . Since this is a free  $\mathbb{Z}$ -module, it is an abelian group. Quotienting out the subgroup H generated by elements of the form

$$(m, (n_1 + n_2)) - (m, n_1) - (m, n_2)$$
  

$$((m_1 + m_2), n) - (m_1, n) - (m_2, n)$$
  

$$(mr, n) - (m, rn)$$

produces the abelian quotient group  $\mathbb{Z}(M \times N)/H$  which we call the tensor product of M and N over R, written  $M \bigotimes_R N$ . We write cosets (m, n) in this abelian group as  $m \otimes n$  and call them simple tensors in the tensor product. Elements of  $M \bigotimes_R N$  are formal  $\mathbb{Z}$ -linear sums of simple tensors.

Note that quotienting out by that particular subgroup basically enforces the following relations (which we write with tensor notation now):

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$
  

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$
  

$$mr \otimes n = m \otimes rn$$

Using these relations we can look at the following examples:

**Example 2.** For any tensor product  $M \bigotimes_R N$  we have  $m \otimes 0 = m \otimes (0+0) = m \otimes 0 + m \otimes 0$ , so  $m \otimes 0 = 0$ . Similarly,  $0 \otimes n = (0+0) \otimes n = 0 \otimes n + 0 \otimes n$ , so  $0 \otimes n = 0$ .

**Example 3.**  $\mathbb{Z}/n \bigotimes_R \mathbb{Z}/m = 0$  whenever n, m are relatively prime. This is because since n, m are relatively prime, for any  $a \in \mathbb{Z}/n$ , ma = a, so for any  $a \in \mathbb{Z}/n$ ,  $b \in \mathbb{Z}/m$ ,  $a \otimes b = ma \otimes b = a \otimes mb = a \otimes 0 = 0$ .

More examples to come...

### Universal Property of Tensor Products

There is a canonical map  $\iota: M \times N \to M \bigotimes_R N$  defined by  $(m, n) \mapsto m \otimes n$ .

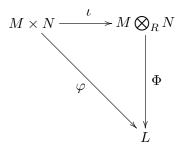
**Definition** (*R*-balanced map). Let M be a right *R*-module, N be a left *R*-module and L be an abelian group. Then a map  $\varphi : M \times N \to L$  is *R*-balanced if it is linear in each variable and additionally,  $\varphi(mr, n) = \varphi(m, rn)$  for all  $m \in M, n \in N, r \in R$ .

So the canonical map  $\iota$  is *R*-balanced.

We now have the analogous universal property of the tensor product:

**Proposition 2** (Universal Property of the Tensor Product). Let M be a right R-module, N be a left R-module and L be any abelian group. Then

there is a bijection between R-balanced maps  $\varphi : M \times N \to L$  and group homomorphisms  $\Phi : M \bigotimes_R N \to L$  that satisfies the commutative triangle:



Proof. In the first direction, let  $\Phi$  be a group homomorphism from  $M \bigotimes_R N \to L$ . Then defining  $\varphi = \Phi \circ \iota$ , we have a map from  $M \times N \to L$ . We need to check that  $\varphi$  is in fact *R*-balanced. Let  $m_1, m_2 \in M$  and  $n \in N$ . Then  $\varphi(m_1+m_2,n) = \Phi(\iota(m_1+m_2,n)) = \Phi((m_1+m_2)\otimes n) = \Phi(m_1\otimes n+m_2\otimes n)$ . Since  $\Phi$  is a group homomorphism, we have  $\Phi(m_1 \otimes n + m_2 \otimes n) = \Phi(m_1 \otimes n) + \Phi(m_2 \otimes n) = \Phi(\iota(m_1,n)) + \Phi(\iota(m_2,n)) = \varphi(m_1,n) + \varphi(m_2,n)$ . So  $\varphi$  is linear in *M*. Similarly we can show that  $\varphi$  is linear in *N*. Let  $r \in R$ , then  $\varphi(mr,n) = \Phi(\iota(mr,n)) = \Phi(mr \otimes n) = \Phi(m \otimes rn) = \Phi(\iota(m,rn)) = \varphi(m,rn)$ . So  $\varphi$  is *R*-balanced in  $M \times N$ .

In the other direction, using Proposition 1 the *R*-balanced map  $\varphi$  defines a  $\mathbb{Z}$ -module homomorphism  $\varphi'$  from the free  $\mathbb{Z}$  module  $\mathbb{Z}(M \times N)$  to *L* such that  $\varphi'(m,n) = \varphi(m,n)$ . Since  $\varphi$  is *R*-balanced,  $\varphi'$  maps elements of the form that generated the subgroup *H* in our definition of the tensor product to 0, so the kernel of  $\varphi'$  contains *H*. Hence,  $\varphi'$  induces a group homomorphism  $\Phi$  on the quotient group  $M \bigotimes_R N$  to *L* by  $\Phi(m \otimes n) = \varphi'(m,n) = \varphi(m,n)$ . Since the elements  $m \otimes n$  generate  $M \bigotimes_R N$ ,  $\Phi$  is uniquely determined by this equation.  $\Box$ 

We now have to consider the module structure of the tensor product. Notice that if R is a commutative ring, then since rm = mr,  $M \bigotimes_R N$  is a left R-module given by:

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn)$$

In this case, Proposition 2 gives a bijection between R-bilinear maps  $M \times N \to L$  (no longer needed to be R-balanced because of this commutativity relation above) and R-module homomorphisms  $M \bigotimes_R N \to L$ .

We have the following fact about the tensor product:

**Proposition 3** (Tensor Product is Associative).  $X \bigotimes (Y \bigotimes Z) \cong (X \bigotimes Y) \bigotimes Z$ .

We actually have two proofs of this fact. The first uses only the universal property of tensor products, while the second uses an application of the Yoneda Lemma (see [Liu18]).

Proof using Universal Property. For each  $x \in X$ , we define a map

$$\phi: Y \times Z \to (X \bigotimes Y) \bigotimes Z$$
$$(y, z) \mapsto (x \otimes y) \otimes z$$

This map is bilinear because  $\phi(y_1 + y_2, z) = (x \otimes (y_1 + y_2)) \otimes z = ((x \otimes y_1) + (x \otimes y_2)) \otimes z = (x \otimes y_1) \otimes z + (x \otimes y_2) \otimes z$  and similarly for the z-coordinate.

Since  $\phi$  is bilinear, by the universal property it induces a linear map

$$\Phi_x: Y \bigotimes Z \to (X \bigotimes Y) \bigotimes Z$$
$$y \otimes z \mapsto (x \otimes y) \otimes z$$

Now we define a map

$$\delta: X \times (Y \otimes Z) \to (X \bigotimes Y) \bigotimes Z$$
$$(x, \sum_{i=1}^{n} y_i \otimes z_i) \mapsto \Phi_x(\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i$$

Again, this map is bilinear (because of linearity of  $\Phi_x$  and properties of the tensor product), so by the universal property it induces a linear map

$$\Delta: X \bigotimes (Y \otimes Z) \to (X \bigotimes Y) \bigotimes Z$$
$$x \otimes (\sum_{i=1}^{n} y_i \otimes z_i) = \sum_{i=1}^{n} x \otimes (y_i \otimes z_i) \mapsto \sum_{i=1}^{n} (x \otimes y_i) \otimes z_i$$

We construct the inverse by fixing  $z \in Z$  and proceeding in a similar way to get

$$\Gamma : (X \bigotimes Y) \otimes Z \to X \bigotimes (Y \bigotimes Z)$$
$$\sum_{i=1}^{n} (x_i \otimes y_i) \otimes z \mapsto \sum_{i=1}^{n} x_i \otimes (y_i \otimes z)$$

Notice that in  $\Delta$  there is only one x and many z and in  $\Gamma$  there is only one z and many x. One might think that this is a problem but we check that  $\Delta, \Gamma$  are indeed mutually inverse:

$$\Gamma(\Delta(\sum_{i=1}^{n} x \otimes (y_i \otimes z_i))) = \Gamma(\sum_{i=1}^{n} (x \otimes y_i) \otimes z_i)$$
$$= \sum_{i=1}^{n} \Gamma((x \otimes y_i) \otimes z_i) \quad \text{By linearity of } \Gamma$$
$$= \sum_{i=1}^{n} x \otimes (y_i \otimes z_i)$$

and

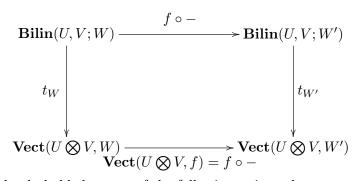
$$\Delta(\Gamma(\sum_{i=1}^{n} (x_i \otimes y_i) \otimes z)) = \Delta(\sum_{i=1}^{n} x_i \otimes (y_i \otimes z))$$
$$= \sum_{i=1}^{n} \Delta(x_i \otimes (y_i \otimes z)) \quad \text{By linearity of } \Delta$$
$$= \sum_{i=1}^{n} (x_i \otimes y_i) \otimes z$$

Proof using Yoneda Lemma. By the Yoneda Lemma, if two representable functors  $H^A, H^{A'}$  are isomorphic, then that means  $A \cong A'$ . We first show that the functor

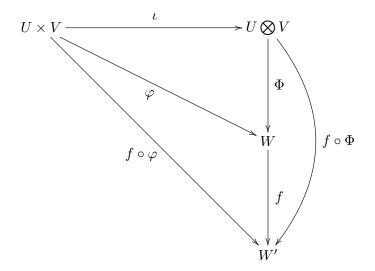
 $\begin{aligned} \mathbf{Bilin}(U,V;-): \mathbf{Vect}_k \to \mathbf{Set} \\ \text{Vector space } W \mapsto \text{set of bilinear maps } U \times V \to W \end{aligned}$ 

is representable by showing that it is naturally isomorphic to the functor  $H^{U \bigotimes V} = \mathbf{Vect}(U \bigotimes V, -)$ 

We define the mapping  $t_W$ :  $\operatorname{Bilin}(U, V; W) \to \operatorname{Vect}(U \otimes V, W)$  by  $t_W(\varphi) = \Phi$  where  $\Phi$  is the linear map out of the tensor product uniquely determined by  $\varphi$  given by the universal property of tensor products. We define  $t_W^{-1}$ :  $\operatorname{Vect}(U \otimes V, W) \to \operatorname{Bilin}(U, V; W)$  by  $t_W^{-1}(\Phi) = \varphi$ . Clearly  $t_W, t_W^{-1}$  are inverse to each other. Let  $f: W \to W'$ . We need to show that the following naturality square holds:



This clearly holds because of the following universal property:



So we have the following chain of isomorphisms:

$$\mathbf{Vect}(X\bigotimes(Y\bigotimes Z), -) \cong \mathbf{Bilin}(X, Y\bigotimes Z; -)$$
$$\cong \mathbf{3\text{-lin}}(X, Y, Z; -)$$
$$\cong \mathbf{Bilin}(X\bigotimes Y, Z; -)$$
$$\cong \mathbf{Vect}((X\bigotimes Y)\bigotimes Z, -)$$

which will give us  $X \bigotimes (Y \bigotimes Z) \cong (X \bigotimes Y) \bigotimes Z$ . The only line in here we need to prove is the isomorphism  $\operatorname{Bilin}(X, Y \bigotimes Z; -) \cong \operatorname{3-lin}(X, Y, Z; -)$ . So we need to establish a bijection between bilinear maps  $X \times (Y \bigotimes Z) \to W$  with 3-linear maps  $X \times Y \times Z \to W$ . Firstly, given a bilinear map  $\varphi: X \times (Y \bigotimes Z) \to W$ , we can define a 3-linear map  $\tilde{\varphi}: X \times Y \times Z \to W$  by  $(x, y, z) \mapsto \varphi(x, \iota(y, z))$  where  $\iota$  is the canonical map from  $Y \times Z \to Y \bigotimes Z$ .  $\tilde{\varphi}$  is clearly 3-linear because  $\varphi$  was linear in x and separately linear in  $\iota(y, z)$ , while  $\iota$  is bilinear.

Now going the other way suppose we had a 3-linear map  $\varphi : X \times Y \times Z \to W$ . How do we get a bilinear map out of  $X \times (Y \bigotimes Z)$ ? Fixing  $x \in X$ , we have the bilinear map out of  $\varphi_x : Y \times Z \to W$  by  $(y, z) \mapsto \varphi(x, y, z)$ . By the universal property of the tensor product,  $\varphi_x$  gives us a linear map  $\delta_x : Y \bigotimes Z \to W$ . We now define the bilinear map  $\hat{\varphi} : X \times (Y \bigotimes Z) \to W$  by  $(x, \sum_{i=1}^n y_i \otimes z_i) \mapsto \delta_x (\sum_{i=1}^n y_i \otimes z_i)$ . Notice that this is a very similar construction in our first proof using the universal property. Since  $\delta_x$  is linear,  $\hat{\varphi}$  is clearly linear in the second argument. It remains to check that  $\hat{\varphi}$  is linear in x:

$$\begin{split} \hat{\varphi}(x_1 + x_2, \sum_{i=1}^n y_i \otimes z_i) &= \delta_{x_1 + x_2} (\sum_{i=1}^n y_i \otimes z_i) \\ &= \sum_{i=1}^n \delta_{x_1 + x_2} (y_i \otimes z_i) \\ &= \sum_{i=1}^n \delta_{x_1 + x_2} (\iota(y_i, z_i)) \\ &= \sum_{i=1}^n \varphi_{x_1 + x_2} (y_i, z_i) \quad \text{By definition of } \delta_x \\ &= \sum_{i=1}^n \varphi(x_1 + x_2, y_i, z_i) \quad \text{By definition of } \varphi_x \\ &= \sum_{i=1}^n \varphi(x_1, y_i, z_i) + \varphi(x_2, y_i, z_i) \quad \text{By trilinearity of } \varphi \end{split}$$

Finally, it turns out that  $\hat{\tilde{\varphi}} = \varphi$  and  $\tilde{\tilde{\varphi}} = \varphi$ , so these are mutually inverse to each other.

# References

[Lei14] Tom Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.