On the asymptotic behavior of the magnitude function for odd-dimensional Euclidean balls

Master's Thesis

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Table of Contents

Table of Contents

Notions of size

- Cardinality of a set
- Measure of a set
- Order of a group
- Volume
- Euler characteristic of a topological space

Origins in category theory

- Leinster in [Lei06] defines the Euler characteristic of a finite category.
- This definition is extended to finite enriched categories.
- Lawvere in [Law73] observed that categories enriched in $[0, \infty]$ can be seen as metric spaces.
- Definition of Euler characteristic for finite categories enriched in $[0, \infty]$ gives the definition of the magnitude of a finite metric space.

Magnitude of a matrix I

Definition 1

Let $M \in M_n(\mathbb{R})$ be a $n \times n$ matrix. A **weighting** on M is a column vector $w \in \mathbb{R}^n$ such that $Mw = 1$. A **coweighting** on M is a row vector $v \in \mathbb{R}^n$ such that $vM = 1^*$.

Magnitude of a matrix II

Lemma 2

Suppose M possesses a weighting w and a coweighting v. Then $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} v_j$.

Proof.

$$\sum_{j=1}^{n} w_j = 1^* w = vMw = v1 = \sum_{j=1}^{n} v_j.$$

$\square$

Magnitude of a matrix III

Definition 3

Let $M \in M_n(\mathbb{R})$. If M possesses a weighting w and coweighting v, then we say M has **magnitude** and its magnitude is given by

$$|M| = \sum_{j=1}^{n} w_j = \sum_{j=1}^{n} v_j.$$

The weighting of a matrix is not guaranteed to exist, but if the matrix is invertible, then the calculation is more straightforward, as the following lemma shows.

Magnitude of a matrix IV

Lemma 4

If M is invertible, then M possesses a unique weighting and its magnitude is given by

$$|M| = \sum_{i,j=1}^{n} \left[M^{-1}\right]_{ij}.$$

Magnitude of a matrix V

Proof.

Suppose M is invertible. Then the equation $Mw = 1$ has

$$w_j = \left[M^{-1}1\right]_j = \sum_{i=1}^{n} m_i^{(j)}$$

where $m^{(j)}$ is the j-th row of M^{-1}, as the unique solution. Then the magnitude of M is given by

$$|M| = \sum_{j=1}^{n} w_j = \sum_{j=1}^{n}\sum_{i=1}^{n}\left[M^{-1}\right]_{ij}.$$

$\square$

Finite metric spaces I

Definition 5

Let A be a finite metric space with distance function d. Define its **similarity matrix** Z_A by

$$[Z_A]_{a,b} = e^{-d(a,b)}.$$

Then if Z_A has magnitude, then we say A has **magnitude** and we write

$$|A| = |Z_A|.$$

Finite metric spaces II

Example 6

1. Let A be the discrete space, that is, for all $a \neq b \in A$, $d(a,b) = \infty$. Let A have n points. Then the similarity matrix of A is I_n, the $n \times n$ identity matrix, and the magnitude $|A| = n$.
2. Let A consist of two points of distance d apart. Then

$$Z_A = \begin{bmatrix} 1 & e^{-d} \\ e^{-d} & 1 \end{bmatrix}$$

and by Lemma 4 above, the magnitude of A is $|A| = 1 + \tanh\left(\frac{d}{2}\right)$.

The magnitude function I

Want to consider the magnitude of a space as we scale distances.

Definition 7

Let $t > 0$. Denote by tA the metric space containing the same points as A but all distances are scaled by t. Then we call the assignment $t \mapsto |tA|$ the **magnitude function** of A.

Note that Z_{tA} might not possess a weighting for all t, so the magnitude function might only be a partially defined function of t.

The magnitude function II

Example 8

Recalling the second example above, suppose A has two points of distance d apart. Then

$$Z_{tA} = \begin{bmatrix} 1 & e^{-td} \\ e^{-td} & 1 \end{bmatrix}$$

and the magnitude function is

$$\mathrm{Mag}(tA) = 1 + \tanh\left(\frac{td}{2}\right).$$

The magnitude function III

Theorem 9 (Proposition 2.2.6 of [Lei11])

Let A be a finite metric space. Then

- *tA is invertible and hence has magnitude for all but finitely many $t > 0$.*
- *The magnitude function of A is analytic at all $t > 0$ such that tA is invertible.*
- *for $t \gg 0$, there is a unique, positive, weighting on tA.*
- *For $t \gg 0$, the magnitude function of A is increasing.*
- *$|tA| \to \#A$ as $t \to \infty$.*

Idea of proof.

As $t \to \infty$, the similarity matrix $Z_{tA} \to I_n$.

The magnitude function IV

Example 10

Let $K_{3,2}$ denote the bipartite graph of 3 and 2 vertices with each edge having distance t and using the shortest-path metric.

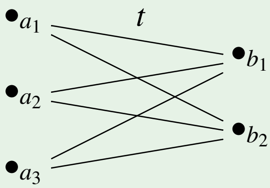

Figure: The graph $K_{3,2}$.

Then the magnitude function of $K_{3,2}$ is given by:

$$\mathrm{Mag}\left(tK_{3,2}\right) = \frac{5-7e^{-t}}{(1+e^{-t})(1-2e^{-2t})}.$$

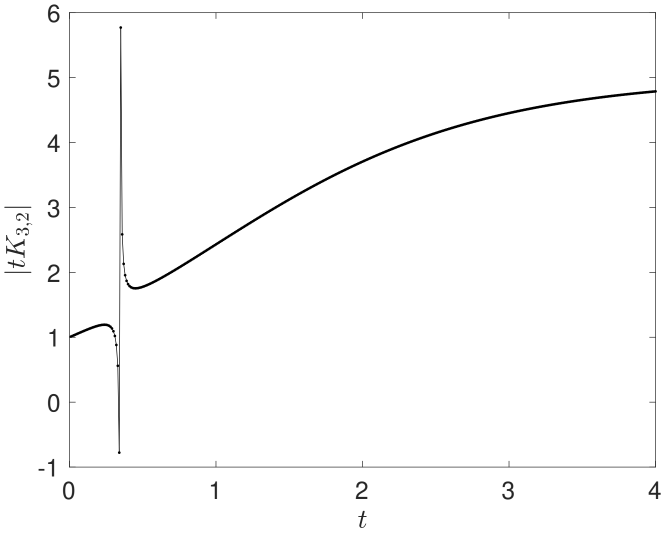

Figure: The magnitude function of $K_{3,2}$.

Positive definite metric spaces I

For what kinds of spaces does magnitude behave more nicely?

Definition 11

A finite metric space A is called **positive definite** if its similarity matrix Z_A is a positive definite matrix.

- Positive definite matrices are always invertible, so any positive definite metric space automatically has magnitude.
- Principal submatrices of positive definite matrices are positive definite so subspaces of positive definite metric spaces also have magnitude.
- Finite subsets of Euclidean space are positive definite metric spaces.

Positive definite metric spaces II

Theorem 12 (Proposition 2.4.3 of [Lei11])

If A is a finite positive definite metric space of n points, then A has magnitude and the magnitude is given by

$$|A| = \sup_{v \neq 0} \frac{\left(\sum_{a\in A} v_a\right)^2}{v^* Z_A v}$$

where $v \in \mathbb{R}^n$. The supremum is attained if and only if v is a nonzero scalar multiple of the unique weighting on A.

Idea of proof.

Cauchy-Schwarz inequality applied to arbitrary v and weighting w.

Positive definite metric spaces III

Theorem 13 (Lemma 2.4.10 of [Lei11])

If A is a positive definite metric space and $B \subseteq A$, then $|B| \leq |A|$.

Proof.

Since A is positive definite and B is a subspace, B is also positive definite and both sapces have magnitude. Furthermore, using Theorem 12 above, we have

$$|B| = \sup_{v \neq 0} \frac{\left(\sum_{b \in B} v_b\right)^2}{v^* Z_B v} \leq \sup_{v \neq 0} \frac{\left(\sum_{a \in A} v_a\right)^2}{v^* Z_B v} = |A|$$

since the space of vectors we are considering for B is a subspace of the space of vectors in consideration for A. □

Infinite metric spaces I

- Want to extend the definition of magnitude to include infinite sets.
- Idea: approximate by finite subsets.
- Question: Is the answer we get via approximation by finite subspaces independent of the approximation we choose?
- Answer: Yes, for a specific class of metric spaces.

Infinite metric spaces II

Definition 14

Let A be a metric space. We say A is a **positive definite metric space** if every finite subspace of A is positive definite. If A has the property that tA is positive definite for all $t > 0$, then A is of **negative type**.

- Negative type is a classical property from metric space theory.
- Many familiar metric spaces are of negative type and we will see later that they have magnitude.

Infinite metric spaces III

Definition 15 (The Hausdorff metric)

Let X be a metric space and A, B be compact subsets. Then $d_H(A,B)$, the **Hausdorff distance** between A, B is defined to be

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}.$$

Infinite metric spaces IV

Definition 16 (Gromov-Hausdorff distance)

Let A, B be compact metric spaces. Then $d_{GH}(A,B)$ the **Gromov-Hausdorff distance** between A, B is defined to be

$$d_{GH}(A,B) = \inf d_H(\varphi(A), \psi(B))$$

where the infimum is over all metric spaces X and isometric embeddings $\varphi : A \to X$ and $\psi : B \to X$.

Infinite metric spaces V

Theorem 17 (Proposition 3.1 of [LM17])

The quantity

$$M(A) = \sup\{|A'| : A' \subseteq A, A' \textit{ finite}\}$$

is a lower semicontinuous function of A (taking values in $[0,\infty]$) in the class of compact positive definite metric spaces equipped with the Gromov-Hausdorff topology.

- Note that if A is a finite positive definite metric space, then by semicontinuity and Theorem 13, $M(A) = |A|$, so this agrees with the finite case.

Infinite metric spaces VI

Definition 18

Let A be a compact positive definite metric space. Then the **magnitude** of A, written $|A|$, is defined to be the value $M(A)$ from above.

Infinite metric spaces VII

Here are some examples of metric spaces that are of negative type and hence their compact and finite subspaces have magnitude:

Example 19

1. ℓ_p^n for $n \geq 1$ and $1 \leq p \leq 2$,
2. $L_p[0,1]$ for $1 \leq p \leq 2$,
3. n-spheres with the geodesic distance ([Wil14]).
4. weighted trees.

Table of Contents

The convex magnitude conjecture I

Computer calculations by Leinster and Willerton in [LW13] led to what became known as the convex magnitude conjecture. It has since been shown to be false, but it, along with various asymptotic versions of the conjecture, motivates many of the results in this section.

Definition 20

Let $\mathscr{K}^n$ be the space of compact convex sets in $\mathbb{R}^n$. A nonempty set in $\mathscr{K}^n$ is called a **convex body**. A function $P : \mathscr{K}^n \to \mathbb{R}$ is called a **valuation** if

- $P(\emptyset) = 0$,
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $A, B, A \cup B \in \mathscr{K}^n$.

The convex magnitude conjecture II

Theorem 21 (Hadwiger's Theorem ([Sch14]))

There are valuations $V_0, V_1, \ldots, V_n$ where each V_i is homogeneous of degree i such that if P is a valuation that is invariant under rigid motions and continuous with respect to the Hausdorff metric, then there are constants $c_0, c_1, \ldots, c_n$ such that

$$P = \sum_{i=0}^{n} c_i V_i.$$

- Each V_i is called the i-th intrinsic volume.
- V_n is the usual n-dimensional volume, V_{n-1} is half the surface area, and V_0 is the Euler characteristic.

The convex magnitude conjecture III

Leinster and Willerton in [LW13] stated the following conjecture.

Convex magnitude conjecture

Let $K \in \mathscr{K}^n$. Then magnitude is a valuation and moreover

$$\mathrm{Mag}(tK) = \sum_{i \geq 0} \frac{V_i(K)}{i!\,\omega_i} t^i$$

where ω_i is the volume of the i-th dimensional unit ball.

The convex magnitude conjecture IV

Theorem 22 (Theorem 7 of [LW13])

The magnitude of the straight line segment L of length ℓ has the form

$$|L| = 1 + \frac{\ell}{2}$$

That is,

$$\operatorname{Mag}\left(tB_2^1\right) = \left|[-t,t]\right| = 1 + t.$$

The convex magnitude conjecture V

Barcelo and Carbery explicitly calculated the magnitude function for odd-dimensional Euclidean balls, showing that the magnitude function is a *rational* function in t:

Theorem 23 ([BC16])

Let d be odd and denote by B_2^d the closed d-dimensional Euclidean ball. Then

$$Mag\left(tB_2^3\right) = \frac{t^3}{3!} + t^2 + 2t + 1,$$

$$Mag\left(tB_2^5\right) = \frac{t^6 + 18t^5 + 135t^4 + 525t^3 + 1080t^2 + 1080t + 360}{5!(t+3)}.$$

The convex magnitude conjecture VI

- In particular, the magnitude function for B_2^5 shows that the convex magnitude conjecture is in general false.
- However, various asymptotic versions of the convex magnitude conjecture (with somewhat different coefficients) have been shown.

Asymptotic results I

Theorem 24 (Theorem 1 of [BC16])

Let K be a nonempty compact set in $\mathbb{R}^n$. Then

$$Mag(tK) \to 1 \text{ as } t \to 0$$

and

$$t^{-n} Mag(tK) \to \frac{Vol(K)}{n!\omega_n} \text{ as } t \to \infty.$$

So the first and last coefficients were correctly predicted by the conjecture as we take $t \to 0$ and $t \to \infty$ respectively.

Asymptotic results II

Theorem 25 (Theorem 2(c)-(d) of [GG17])

Let $d \geq 3$ be odd and let K be a d-dimensional convex body with nonempty interior and smooth boundary. Then

$$Mag(tK) = \frac{1}{d!\omega_d}\Big(V_d(K)t^d + (d+1)V_{d-1}(K)t^{d-1} + \frac{\pi}{4}(d+1)^2 V_{d-2}(K)t^{d-2}\Big) + O\left(t^{d-3}\right)$$

as $t \to \infty$.

So we can recover the next two intrinsic volumes in the asymptotic expansion for large t, but the coefficients are not the ones predicted by the convex magnitude conjecture.

Asymptotic results III

Theorem 26 (Theorem 4 of [Mec19])

Let $d = 2m+1$ be odd and denote by B_2^d the d-dimensional closed Euclidean ball. Then

$$\frac{d}{dt} Mag\left(tB_2^d\right)\Big|_{t=0} = \frac{1}{2} V_1\left(B_2^d\right)$$

where

$$V_1\left(B_2^d\right) = \frac{(2m+1)\sqrt{\pi}\,\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} = 2\binom{m-\frac{1}{2}}{m}^{-1}$$

is the first intrinsic volume of the closed unit ball.

Asymptotic results IV

- This result depends on work done by Willerton in [Wil17] to reformulate the magnitude function of odd-dimensional Euclidean balls in terms of collections of Schröder paths.

Table of Contents

Schröder paths I

Definition 27

- A **Schröder path** is a finite directed path in the integer lattice in which each step starting at $(x,z) \in \mathbb{Z}^2$ is either an **ascent** to $(x+1, y+1)$, a **descent** to $(x+1, y-1)$ or a **flat step** to $(x+2, y)$.
- Fix $k \geq 0$. A **disjoint** k**-collection** is a family of Shröder paths from $(-i, i)$ to $(i, i$ for each $0 \leq i \leq k$ such that no node in $\mathbb{Z}^2$ is contained in more than one path.
- We denote by X_k the set of all disjoint k-collections and by X_k^j the sete of all disjoint k-collections containing exactly j flat steps.

Schröder paths II

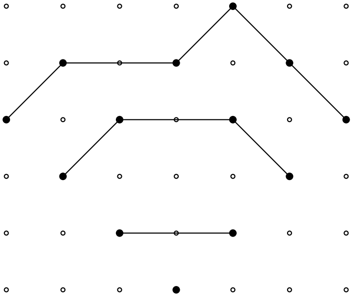

Figure: A disjoint 3-collection.

Schröder paths III

- What does X_k^0 look like? Starting from $i = 0$ and moving up we realize that each path at height i can only consist of i ascents followed by i descents. We will call this kind of path a **V-path at height** i.
- The set X_k^0 consists of only one collection, which we'll call σ_{roof}^k, which is entirely made up of V-paths for each $0 \leq i \leq k$.

Schröder paths IV

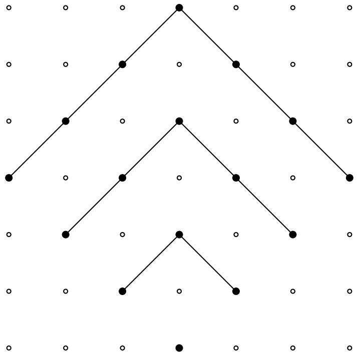

Figure: The disjoint 3-collection σ^3_{roof}.

Schröder paths V

Let $\sigma \in X_k$ be a disjoint k-collection. For each path in σ we associate to each step τ a weighting by

$$\omega_j(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an ascent,} \\ t & \text{if } \tau \text{ is a flat step,} \\ y+1-j & \text{if } \tau \text{ is a descent from height } y \text{ to height } y-1. \end{cases}$$

For a collection $\sigma \in X_k$ the **total weight** of σ, denoted $\omega_j(\sigma)$ is the product of the weightings of all the steps on each path in σ. So $\omega_j(\sigma)$ is a polynomial in t with degree the maximum number of flat steps in σ.

Schröder paths VI

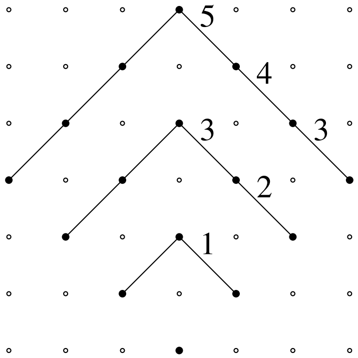

Figure: The disjoint 3-collection σ^3_{roof} with ω_2 weightings.

Schröder paths VII

Theorem 28 (Corollary 27 of [Wil17])

Let $d = 2m+1$ be odd. Then

$$Mag\left(tB_2^d\right) = \frac{\sum\limits_{\sigma \in X_{m+1}} \omega_2(\sigma)}{d! \sum\limits_{\sigma \in X_{m-1}} \omega_0(\sigma)} =: \frac{N(t)}{d! D(t)}$$

for all $t > 0$.

Meckes proved his result in Theorem 26 by differentiating the expression above, evaluating at zero and simplifying.

Table of Contents

The problem I

Goal

Our goal is to calculate the value of

$$\frac{d^2}{dt^2}\mathrm{Mag}\left(tB_2^d\right)\Big|_{t=0}.$$

The problem II

Outline

1. Apply the quotient rule to Theorem 28 and the result of Theorem 26 to express the second derivative in terms of $N(t), D(t)$.
2. The expression will contain first and second derivatives of $N(t)$ and $D(t)$ evaluated at $t = 0$. First consider disjoint k-collections containing exactly one flat step to simplify terms containing first derivatives.
3. Consider disjoint k-collections containing exactly two flat steps to simplify terms containing second derivatives.
4. Combine the previous two steps to arrive at an (partial) answer.

Second derivative

After applying the quotient rule, we get

$$\frac{d^2}{dt^2}\mathrm{Mag}\left(tB_2^d\right)\Big|_{t=0} = 2\left[\frac{N''(0) - d!D''(0)}{N(0)}\right] - V_1\left(B_2^d\right)\left[\frac{D'(0)}{D(0)}\right].$$

where

$$D(0) = \omega_0\left(\sigma_{\mathrm{roof}}^{m-1}\right) = \prod_{k=0}^{m-1}\frac{(2k+1)!}{(k+1)!}$$

$$N(0) = \omega_2\left(\sigma_{\mathrm{roof}}^{m+1}\right) = \prod_{k=0}^{m+1}\frac{(2k-1)!}{(k-1)!}$$

Collections with one flat step I

$$D'(0) = t^{-1} \sum_{\sigma \in X^1_{m-1}} \omega_0(\sigma)$$

So we need to consider disjoint $(m-1)$-collections of Schröder paths containing exactly one flat step.

Collections with one flat step II

- Let $\sigma \in X_k^1$ with path containing flat step at height p.
- All paths below p must be V-paths.
- The disjointness condition means the flat step at height p must be centered.
- Paths above the flat step path M-paths or V-paths.
- Cannot have V-paths below M-paths.

Lemma 29

Let $\sigma_{p,q}^{m-1}$ denote the disjoint $(m-1)$-collection with centered flat step at height p and q M-paths above it. Then

$$X_{m-1}^1 = \bigcup_{\substack{1 \leq p \leq m-1 \\ 0 \leq q \leq m-1-p}} \left\{ \sigma_{p,q}^{m-1} \right\}.$$

Collections with one flat step III

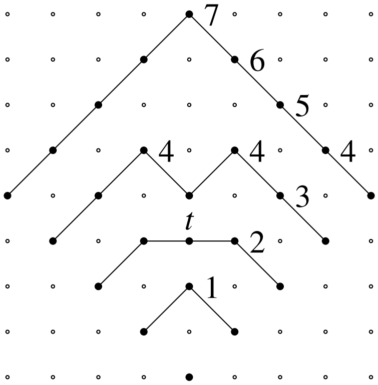

Figure: A disjoint 4-collection with one flat step at height 2 and with ω_2 weightings and total weight $80640t$.

Collections with one flat step IV

By Lemma 29 above, we have

$$D'(0) = \sum_{\substack{1 \le p \le m-1 \\ 0 \le q \le m-1-p}} \left(\prod_{k=0}^{p-1} \frac{(2k+1)!}{(k+1)!} \right) \left(\frac{(2p)!}{(p+1)!} \right) \times \left(\prod_{k=p+1}^{p+q} \frac{(2k)!(2k)}{(k+1)!} \right) \left(\prod_{k=p+q+1}^{m-1} \frac{(2k+1)!}{(k+1)!} \right).$$

So we have

$$\frac{D'(0)}{D(0)} = \sum_{\substack{1 \le p \le m-1 \\ 0 \le q \le m-1-p}} \frac{1}{2p+1} \prod_{k=p+1}^{p+q} \left(\frac{2k}{2k+1} \right).$$

Collections with two flat steps I

- The $N''(0) - d!D''(0)$ term requires us to think about collections containing exactly two flat steps.
- Again, all paths underneath the one containing flat step(s) must be V-paths.
- But then this means that we can't have one path containing two flat steps.
- Furthermore, the path containing the first flat step must have its flat step centered.
- We can have M-paths above the first flat step path.
- Second flat step can either be centered, or offset by one space to the left or right.
- Above an off-centered flat step, we can have **asymmetric M-paths** followed by M-paths and then followed by V-paths.

Collections with two flat steps II

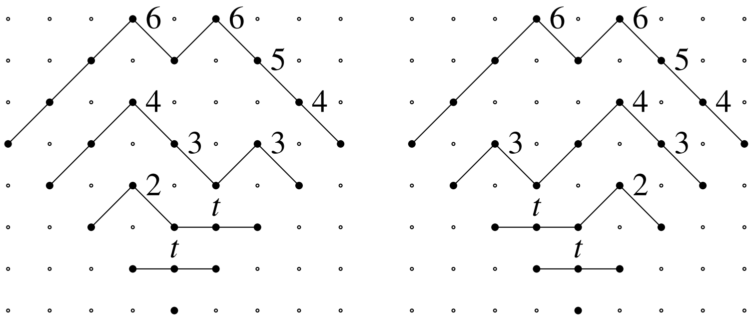

Figure: Disjoint 4-collections containing two flat steps that are reflections of each other. Using ω_2 weightings, both have total weight $51840t^2$.

Collections with two flat steps III

- Denote by $\sigma^k_{p_1,p_2,q_1,q_2}$ the disjoint k-collection with two centered flat step paths at heights p_1 and p_2 respectively, q_1 M-paths above the first flat step, q_2 M-paths above the second flat step, and V-paths everywhere else.
- Denote by $L^k_{p_1,p_2,q_1,q_2}$ the disjoint k-collection with two flat step paths at heights p_1 and p_2 respectively where the flat step at p_2 is offset one to the left, M-paths between p_1 and p_2, q_1 asymmetric (left) M-paths above p_2, q_2 M-paths above p_2+q_1 and V-paths everywhere else.
- $R^k_{p_1,p_2,q_1,q_2}$: The same as $L^k_{p_1,p_2,q_1,q_2}$ but the right offset case.

Then

$$X^2_k = \bigcup_{p_1,p_2,q_1,q_2} \{\sigma^k_{p_1,p_2,q_1,q_2}\} \cup \bigcup_{p_1,p_2,q_1,q_2} \{L^k_{p_1,p_2,q_1,q_2}\} \cup \bigcup_{p_1,p_2,q_1,q_2} \{R^k_{p_1,p_2,q_1,q_2}\}.$$

Collections with two flat steps IV

- $N''(0) - d!D''(0)$ means we will have to consider both X^2_{m-1} as well as X^2_{m+1}.
- Can apply a trick of thinking of collections in X_{m-1} as being embedded in X_{m+1}.

Lemma 30 (Willerton)

Let $d = 2m+1$ and $\sigma \in X_{m-1}$. Then define $\mu(\sigma) \in X_{m+1}$ by shifting all paths in σ up two units, adding ascents from $(-i,i)$ to $(-i+1,i+1)$ and descents from $(i-1,i+1)$ to (i,i) for $1 \leq i \leq m$ and finally adding a V-path at height $m+1$. Then

$$\omega_2(\mu(\sigma)) = d!\omega_0(\sigma).$$

Collections with two flat steps V

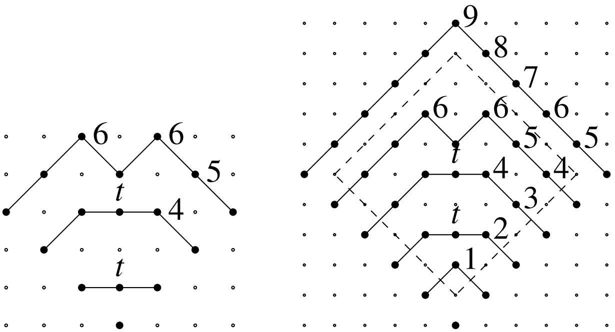

Figure: A disjoint 3-collection σ and its embedding as a disjoint 5-collection. On the left $\omega_0(\sigma) = 720t$ and on the right $\omega_2(\mu(\sigma)) = 261,273,600t = 9! \times 720t$.

Collections with two flat steps VI

- So we only need to consider those paths in X^2_{m+1} that are not μ-embeddings of paths from X^2_{m-1}, that is,

$$N''(0) - d!D''(0) = t^{-2} \sum_{\sigma \in X^2_{m+1} \setminus \mu(X^2_{m-1})} \omega_2(\sigma).$$

Collections with two flat steps VII

There are four disjoint cases:

- The first flat step at height $p_1 = 1$ and the second flat step is at height p_2 where $2 \leq p_2 \leq m+1$.
- The first flat step is at height $p_1 \geq 2$. The second flat step is at height $p_2 = m$ and we either have a M-path or an asymmetric M-path above p_2.
- The first flat step is at height $p_1 \geq 2$. The second flat step is at height $p_2 = m+1$.
- The two flat steps are at heights between 2 and $m-1$ but with no V-paths above height p_2.

Collections with two flat steps VIII

- Actually we need to multiply the number of cases above by two because we need to consider case where the two flat steps are centered (σ) and the case where the second flat step is offset to the left (L).
- Don't need to consider the right offset case by symmetry.
- But actually it turns out that the total weights for both cases is the same! So we only need to consider the σ case.
- After a lot of simplifying, we get...

Partial results I

Let $d = 2m+1$ be odd and B_2^d be the closed d-dimensional Euclidean ball. Then

$$\frac{d^2}{dt^2}\mathrm{Mag}(tB_2^d)\Big|_{t=0} =$$

$$6 \sum_{\substack{2 \le p_2 \le m+1 \\ 0 \le q_1 \le p_2 - 2 \\ 0 \le q_2 \le m+1-p_2}} \frac{1}{2p_2 - 1} \left(\prod_{k=2}^{q_1+2} \frac{(2k-2)}{(2k-1)} \right) \left(\prod_{k=p_2+1}^{p_2+q_2} \frac{(2k-2)}{(2k-1)} \right) +$$

$$6 \sum_{\substack{2 \le p_1 \le m-1 \\ 0 \le q_1 \le m-p_1-1}} \frac{1}{2p_1 - 1} \left(\prod_{k=p_1+1}^{p_1+q_1} \frac{(2k-2)}{(2k-1)} \right) \left(\frac{2m}{(2m-1)(2m+1)} \right) +$$

$$6 \sum_{\substack{2 \le p_1 \le m \\ 0 \le q_1 \le m-p_1}} \frac{1}{2p_1 - 1} \left(\prod_{k=p_1+1}^{p_1+q_1} \frac{(2k-2)}{(2k-1)} \right) \left(\frac{1}{2m+1} \right) +$$

$$6 \sum_{\substack{2 \le p_1 \le m \\ p_1+1 \le p_2 \le m-1 \\ 0 \le q_1 \le p_2-p_1-1}} \frac{1}{2p_1 - 1} \left(\prod_{k=p_1+1}^{p_1+q_1} \frac{(2k-2)}{(2k-1)} \right) \frac{1}{2p_2-1} \left(\prod_{k=p_2+1}^{m+1} \frac{(2k-2)}{(2k-1)} \right) -$$

$$V_1 \sum_{\substack{1 \le p \le m-1 \\ 0 \le q \le m-1-p}} \frac{1}{2p+1} \prod_{k=p+1}^{p+q} \left(\frac{2k}{2k+1} \right).$$

Partial results II

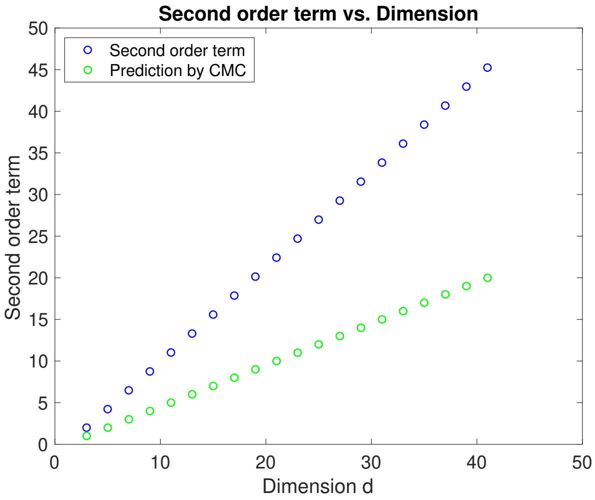

Figure: Second order terms computed for odd dimensions from 3 to 41 along with the values as predicted by the erstwhile convex magnitude conjecture.

Future work

- Continue trying to simplify the long expression for the second derivative term (skip factorials, Catalan numbers,...)

Questions?

Thank you for listening!

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