Adjoints

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Abstract

We should think about adjunctions as an interesting comparison of two categories that is somewhat more general and of a different nature than an equivalence of categories. Following [Lei14], we'll be looking at three different ways of understanding adjoint functors and showing that they are equivalent.

Hom-Set Definition

Definition (Adjoint Functors)**.** Given a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, we say F is left adjoint to G, and G right adjoint to F, written $F \dashv G$ if there is a natural isomorphism $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ for each A in $\mathscr{A}$ and B in $\mathscr{B}$. An adjunction between F and G is a choice of natural isomorphism $t_{A,B}$.

So this means for each $g : F(A) \to B$, we have a map $t_{A,B}(g) : A \to G(B)$. We shall call this isomorphism the transpose of g (Leinster denotes this $\overline{g}$) and this process "transposing" g. Similarly, for each $f : A \to G(B)$, we have a map $t^{-1}_{A,B}(f) : F(A) \to B$.

Naturality

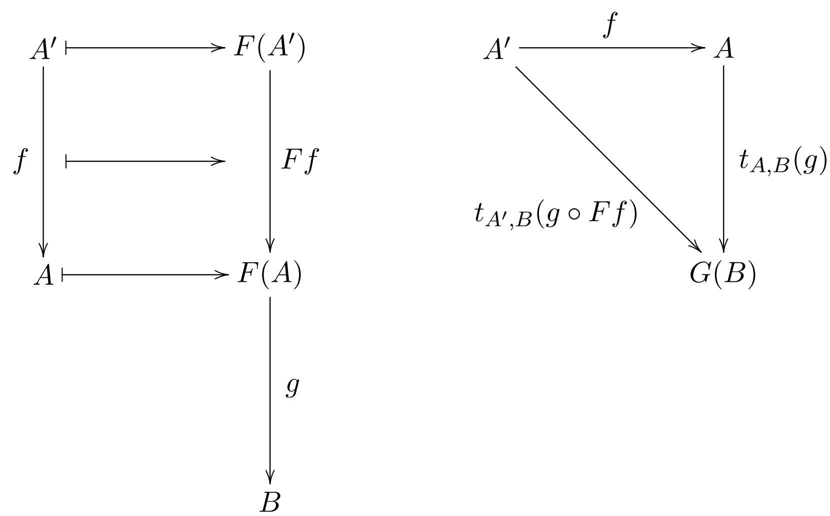
Let's take a closer look at what naturality means. In words it would mean that the transpose of a composition of two maps is equal to the composition of the transpose of the two maps. We have four options here:

1. naturality of t with respect to A
2. naturality of t^{-1} with respect to $F(A)$
3. naturality of t^{-1} with respect to B

4. and finally naturality of t with respect to $G(B)$.

Let's first take a look at *naturality of t with respect to A*:

We have the following data (left), and applying $t_{A',B}$ on $g \circ Ff$ and on them separately we get the commutative triangle on the right:

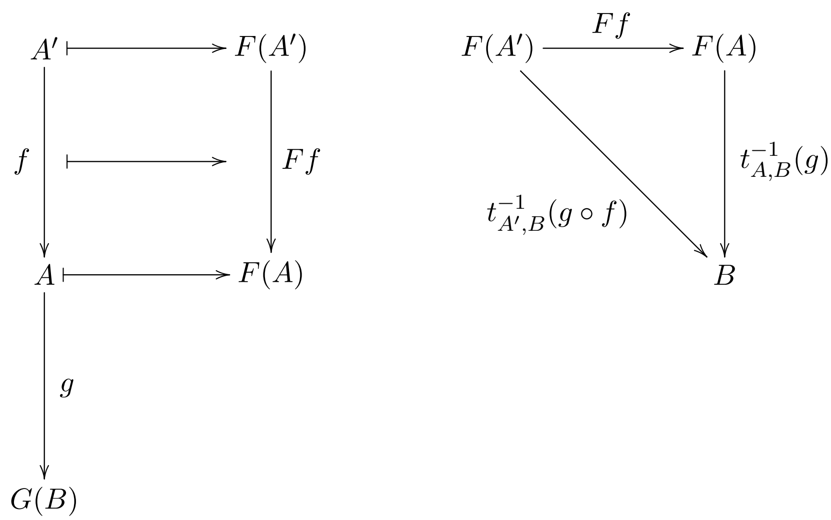


So $t_{A',B}(g \circ F(f)) = t_{A,B}(g) \circ f$ (here $t_{A',F(A)}(F(f)) = f$).

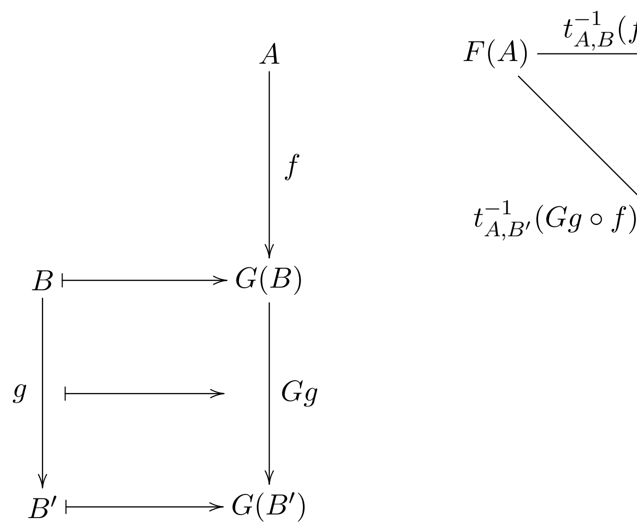
Similarly for 2, 3, and 4, we have the following data yielding the following commutative triangles:

naturality of t^{-1} with respect to $F(A)$:

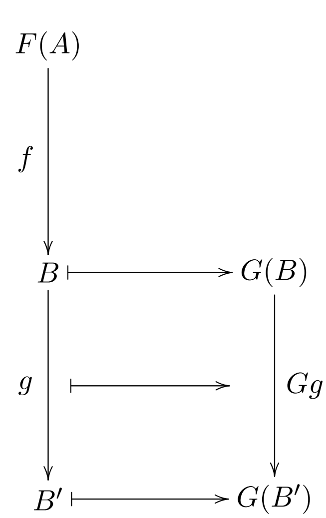
We begin with the map $Ff : F(A') \to F(A)$, and taking the preimage, we get the following data and corresponding commutative triangle:

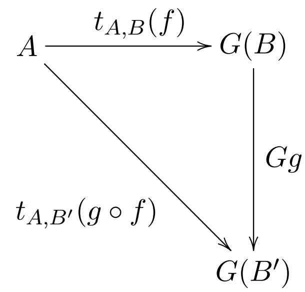


So $t^{-1}_{A',B}(g \circ f) = t^{-1}_{A,B}(g) \circ Ff$.
naturality of t^{-1} *with respect to* B:



So $t^{-1}_{A,B'}(Gg \circ f) = g \circ t^{-1}_{A,B}(f)$.
Finally, *naturality of* t *with respect to* $G(B)$:



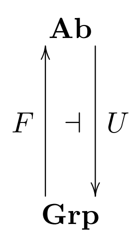


So $t_{A,B'}(g \circ f) = Gg \circ t_{A,B}(f)$.

We call this understanding of adjoint functors the Hom-Set Definition because the important bit here is this isomorphism between the Hom-Sets of $\mathscr{A}$ and $\mathscr{B}$.

There are a whole class of examples of adjoint functors that are the forgetful and free functors between algebraic theories. We'll be looking at one of these:

Example (Abelianization of Groups)**.** There is an adjunction



where U is the forgetful inclusion functor from the category of abelian groups to the category of groups, and F is the free functor from the category of groups to the category of abelian groups. For a group G in **Grp**, $F(G)$ is the abelianization of the group G, or G/G' where G' is the commutator subgroup of G (see my writeup at [Liu18] for details). This abelianization gives rise to the universal property that for any group homomorphism ϕ out of G to an abelian group A, there is a unique $\overline{\phi} : G/G' \to A$ such that

$\phi = \overline{\phi} \circ \pi$ where π is the canonical quotient map from G to G/G'. This universal property is what allows us to specify what $t_{G,A} : \mathbf{Ab}(F(G), A) \to \mathbf{Grp}(G, U(A))$ should do: $t_{G,A}(\overline{\phi}) = \overline{\phi} \circ \pi = \phi$, and $t^{-1}_{G,A}(\phi) = \overline{\phi}$.

Units and Counits Definition

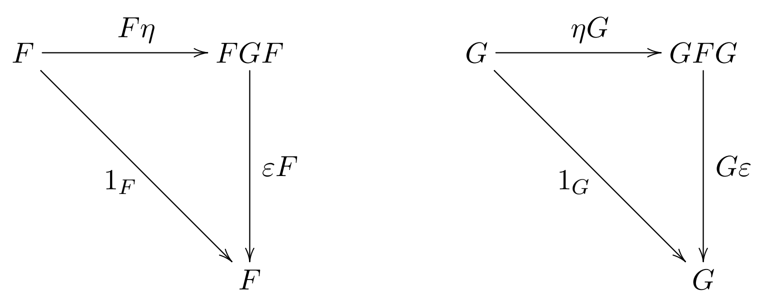
Definition (Unit and Counit of an Adjunction)**.** Given $A \in \mathscr{A}$ and the identity map $1_{F(A)}$, $t_{A,F(A)}(1_{F(A)})$ defines the isomorphism $\eta_A : A \to GF(A)$. Similarly, given $B \in \mathscr{B}$ and the identity map $1_{G(B)}$, $t^{-1}_{G(B),B}(1_{G(B)})$ defines the isomorphism $\varepsilon_B : FG(B) \to B$. Together, η_A and ε_B define the natural transformations

$$\eta : 1_{\mathscr{A}} \to G \circ F, \qquad \varepsilon : F \circ G \to 1_{\mathscr{B}}$$

called the unit and counit of the adjunction, respectively.

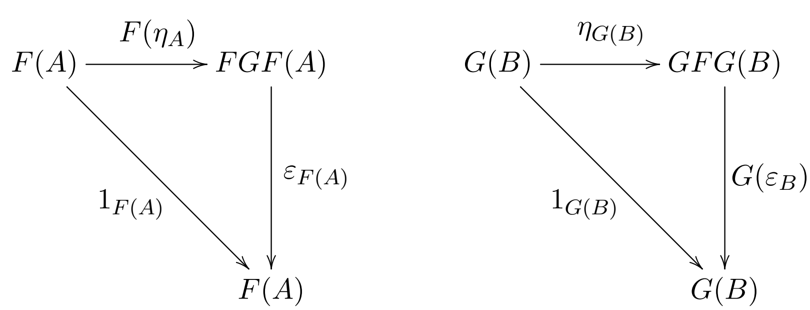
We have important triangle identities associated with the unit and counit.

Proposition 1 (Triangle Identities)**.** *Given an adjunction $F \dashv G$ with unit η and counit ε, the triangles*



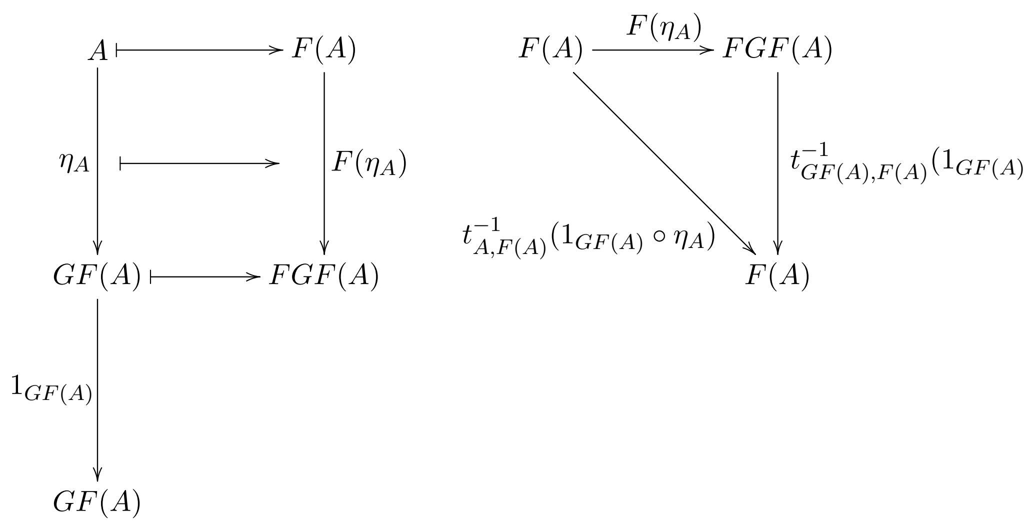
commute.

Proof. We prove the equivalent statement that the triangles



commute for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

For the triangle on the left, we use *naturality of* t^{-1} *with respect to* $F(A)$ that we explained above where we replace f with η_A and g with $1_{GF(A)}$. So we have the following data giving rise to the commutative triangle on the right:



Now by definition $t^{-1}_{GF(A),F(A)}(1_{GF(A)}) = \varepsilon_{F(A)}$, and $t^{-1}_{A,F(A)}(1_{GF(A)} \circ \eta_A) = t^{-1}_{A,F(A)}(\eta_A)$ and by definition, $t_{A,F(A)}(1_{F(A)}) = \eta_A$, so $t^{-1}_{A,F(A)}(\eta_A) = 1_{F(A)}$. So from the triangle we get $1_{F(A)} = \varepsilon_{F(A)} \circ F(\eta_A)$, proving the commutative triangle.

Similarly, for the triangle on the right, we use *naturality of* t *with respect to* $G(B)$ that we explained above where we replace f with $1_{FG(B)}$ and g with ε_B. So we from the resulting commutative triangle we have

$$t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = G(\varepsilon_B) \circ t_{G(B),FG(B)}(1_{FG(B)}).$$

And again, by definition $t_{G(B),FG(B)}(1_{FG(B)}) = \eta_{G(B)}$ and $t_{G(B),B}(\varepsilon_B \circ 1_{FG(B)}) = t_{G(B),B}(\varepsilon_B) = 1_{G(B)}$, so $1_{G(B)} = G\varepsilon_B \circ \eta_{G(B)}$, proving the identity. □

It turns out the unit and counit determine the whole adjunction.

Proposition 2. *Given an adjunction* $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ *for any* $g : F(A) \to B$, $t_{A,B}(g) = Gg \circ \eta_A$, *and for any* $f : A \to G(B)$, $t^{-1}_{A,B}(f) = \varepsilon_B \circ Ff$.

Proof. For any $g : F(A) \to B$, by naturality, $t_{A,B}(g) = t_{A,B}(g \circ 1_{F(A)})$ which by naturality of t with respect to $GF(A)$, is equal to $Gg \circ \eta_A$. Similarly, for any $f : A \to G(B)$, $t^{-1}_{A,B}(f) = t^{-1}_{A,B}(1_{G(B)} \circ f)$ which by naturality of t^{-1} with respect to $FG(B)$, is equal to $\varepsilon_B \circ Ff$. □

Using this fact, we can equivalently define adjunctions by specifying pairs of units and counits.

Theorem 1. *Given functors $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$, there is a bijection between adjunctions $F \dashv G$ and pairs of units and counits (η, ε) that satisfy the triangle identities.*

Proof. We have already shown that given an adjunction t, we can define natural transformations η, ε that satisfy the triangle identites. Now, we just need to show that given unit and counit η, ε, we can uniquely define a natural isomorphism $t_{A,B} : \mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$ for all $A \in \mathscr{A}$, for all $B \in \mathscr{B}$.

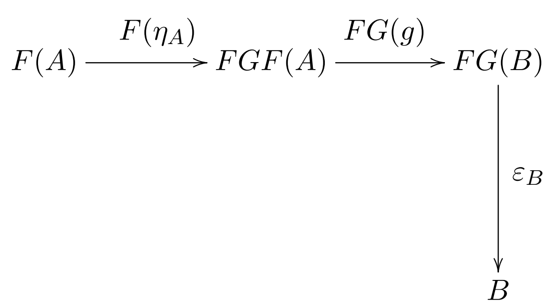
Given $g : F(A) \to B$, define $t_{A,B}(g) = Gg \circ \eta_A$, and given $f : A \to G(B)$, define $t^{-1}_{A,B}(f) = \varepsilon_B \circ Ff$. We need to show that t and t^{-1} are well-defined, mutually inverse, natural, and that η, ε are in fact their unit and counit.

Well-Defined Let $g : F(A) \to B$, $h : F(A) \to B$ with $g = h$. Since G is well-defined, $t_{A,B}(g) = Gg \circ \eta_A = Gh \circ \eta_A = t_{A,B}(h)$, so t is well-defined. Similarly for t^{-1}.

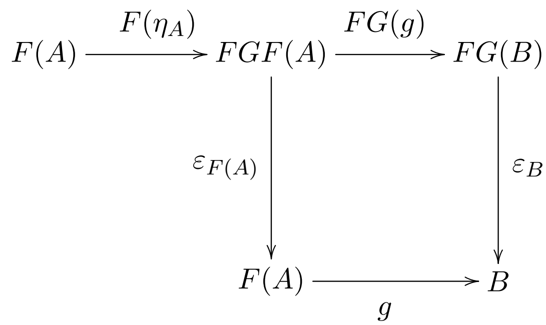
Isomorphism Let $g : F(A) \to B$. Need to show that $t^{-1}_{A,B}(t_{A,B}(g)) = g$. Now by the definition of t and functoriality of F:

$$\begin{aligned} t^{-1}_{A,B}(t_{A,B}(g)) &= t^{-1}_{A,B}(Gg \circ \eta_A) \\ &= \varepsilon_B \circ F(Gg \circ \eta_A) \\ &= \varepsilon_B \circ FGg \circ F(\eta_A) \end{aligned}$$

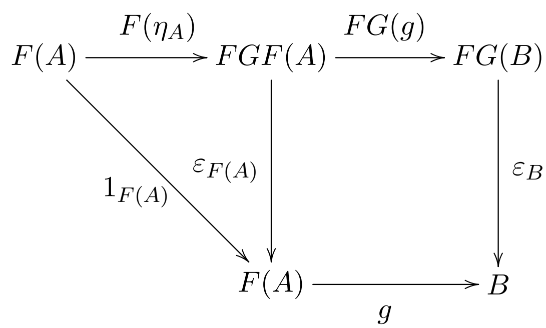
So we have the diagram



However, by naturality of ε with respect to the map g, we can add the following naturality square:



And finally we have the triangle identity, so we can finally add:

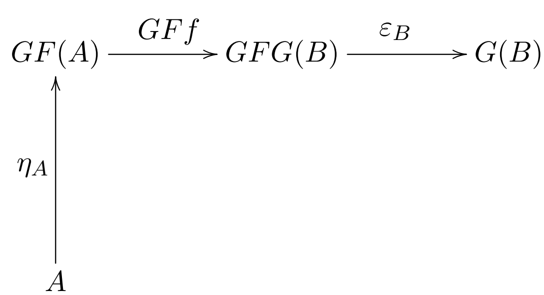


So by commutativity of the diagram, $\varepsilon_B \circ FGg \circ F(\eta_A) = g \circ 1_{F(A)} = g$.

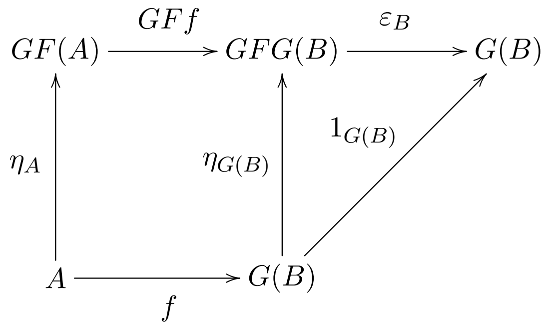
Let $f : A \to G(B)$. We also need to show $t_{A,B}(t^{-1}_{A,B}(f)) = f$. Again by our definition of t and functoriality of G, we have

$$\begin{aligned} t_{A,B}(t^{-1}_{A,B}(f)) &= t_{A,B}(\varepsilon_A \circ Ff) \\ &= G(\varepsilon_B \circ Ff) \circ \eta_A \\ &= G(\varepsilon_B) \circ GFf \circ \eta_A \end{aligned}$$

Again we have the diagram

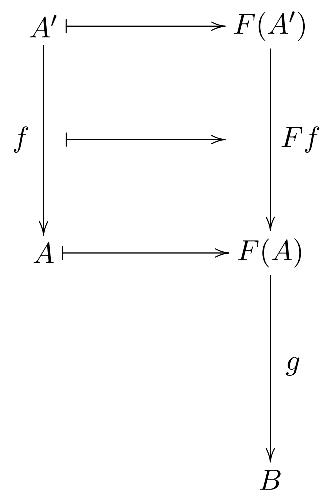


which again, because of naturality of η with respect to the map f and the triangle identity, we have

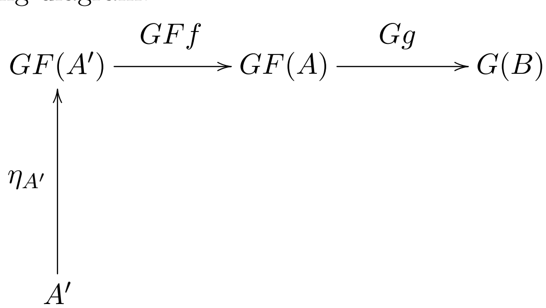


So indeed $G(\varepsilon_B) \circ GFf \circ \eta_A = 1_{G(B)} \circ f = f$.

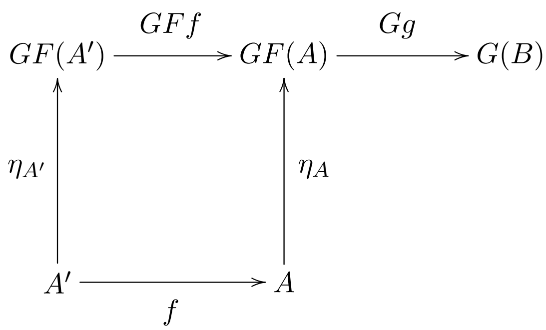
Naturality Now we will show that t is natural in A and that t^{-1} is natural in B. First we have the following data:



and by definition $t_{A',B}(g \circ Ff) = G(g \circ Ff) \circ \eta_{A'} = Gg \circ GFf \circ \eta_{A'}$. We want to show that this equals $Gg \circ \eta_A \circ f = t_{A,B}(g) \circ f$. From $Gg \circ GFf \circ \eta_{A'}$ we have the following diagram:

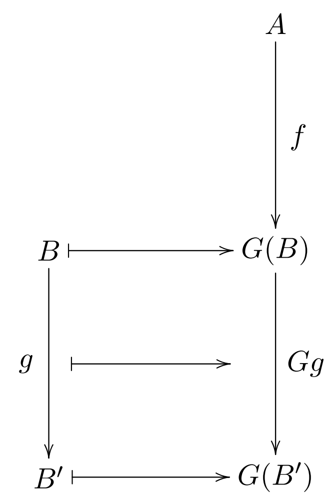


but by naturality of η with respect to the map f, we can add the naturality square:

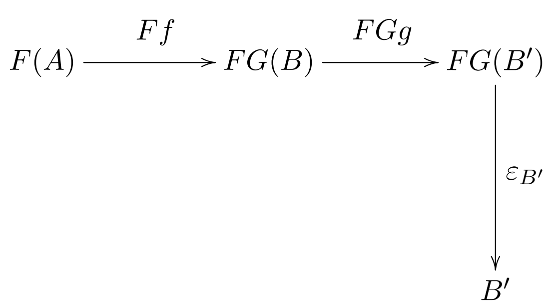


So $Gg \circ GFf \circ \eta_{A'} = Gg \circ \eta_A \circ f = t_{A,B}(g) \circ f$, so t is natural in A.

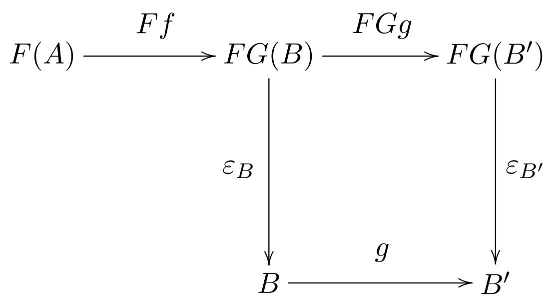
For naturality of t^{-1} in B, we have the following data:



and again by definition of t^{-1} and functoriality of F, we have $t^{-1}_{A,B'}(Gg \circ f) = \varepsilon_{B'} \circ F(Gg \circ f) = \varepsilon_{B'} \circ FGg \circ Ff$. We want to show that this is equal to $g \circ \varepsilon_B \circ Ff$. From $\varepsilon_{B'} \circ FGg \circ Ff$ we have the following diagram:



but again by naturality of ε with respect to the map g, we can add the naturality square:



So indeed $\varepsilon_{B'} \circ FGg \circ Ff = g \circ \varepsilon_B \circ Ff$.

Inverse We finally need to show that this process of corresponding between t and (η, ε) is mutually inverse.

We began with the adjunction $t : \mathscr{B}(F(-), -)) \to \mathscr{A}(-, G(-))$ and from it we derived the unit and counit of the adjunction, namely that $\eta_A = t_{A,FA}(1_{FA})$ and $\varepsilon_B = t^{-1}_{GB,B}(1_{GB})$. So this process defines a mapping $t \mapsto (\eta^t, \varepsilon^t)$. Let's call this mapping Γ, that is, $\Gamma(t) = (\eta^t, \varepsilon^t)$. Going the other way we showed that the unit and counit determine the whole adjunction, meaning that for any $g : FA \to B$, $t_{A,B}(g) = Gg \circ \eta_A$ and for any $f : A \to GB$, $t^{-1}_{A,B}(f) = \varepsilon_B \circ Ff$. So we have a mapping $(\eta, \varepsilon) \mapsto s$. Notice that this unit and counit pair are not necessarily the same ones derived from the adjunction t, and the resulting adjunction from this pair is not necessarily the same as before. Something else to note is that technically from η we get the adjunction s, while ε would give us s^{-1}. But we're just going to worry about s for now. Let's call this mapping Δ, that is, $\Delta(\eta, \varepsilon) = s$. The goal is to then show that $\Delta(\Gamma(t)) = t$ and $\Gamma(\Delta(\eta, \varepsilon)) = (\eta, \varepsilon)$.

First we show $\Delta(\Gamma(t)) = t$. Now I'm fudging with the notation a little bit here, but for all A and B, $\Gamma(t_{A,B}) = (t_{A,FA}(1_{FA}), t^{-1}_{GB,B}(1_{GB}))$. And $\Delta(t_{A,FA}(1_{FA}), t^{-1}_{GB,B}(1_{GB})) = Gg \circ t_{A,FA}(1_{FA})$ for any $g : FA \to B$ (Notice we threw away t^{-1} here, like we noted above). Now by naturality of t, $Gg \circ t_{A,FA}(1_{FA}) = t_{A,B}(g \circ 1_{FA}) = t_{A,B}(g)$. (The same thing happens to t^{-1} by naturality). So indeed $\Delta(\Gamma(t)) = t$.

Finally we show $\Gamma(\Delta(\eta, \varepsilon)) = (\eta, \varepsilon)$. Now again, not worrying particularly about inverses and fudging with the notation, we have $\Delta(\eta, \varepsilon)) = G(-) \circ \eta_A$. And $\Gamma(G(-) \circ \eta_A) = G1_{FA} \circ \eta_A = 1_{GFA} \circ \eta_A = \eta_A$. A similar thing happens with $\varepsilon_B \circ F(-)$. So in fact $\Gamma(\Delta(\eta, \varepsilon)) = (\eta, \varepsilon)$.

So the processes are mutually inverse, concluding the proof. □

References

[Lei14] Tom Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, United Kingdom, 2014.

[Liu18] Stephen Liu. Abelianization of groups, 2018. https://ssyl55.github.io/files/abelianization.pdf.