

Abelianization of Groups

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Abstract

Some notes on how to “abelianize” a group G , and a universal property that arises from this process.

Definition (Commutator). Let G be a group and $x, y \in G$. Then the commutator of x and y , is $[x, y] = x^{-1}y^{-1}xy$.

It is easy to see from this definition that $xy = yxx^{-1}y^{-1}xy = yx[x, y]$ and that $xy = yx$ if and only if $[x, y] = 1$.

Definition (Commutator Subgroup). Suppose A, B are nonempty subsets of G , then $[A, B] = \langle [a, b] | a \in A, b \in B \rangle$ is the group generated by commutators of elements from A and from B . In particular, $G' = [G, G] = \langle [x, y] | x, y \in G \rangle$ is called the commutator subgroup of G .

This commutator subgroup has nice properties which allow us to “abelianize” G , that is, get an abelian group from G . Since we saw above that x, y commute if and only if $[x, y] = 1$, this suggests that to get an abelian group from G , we need some way of *setting* $[x, y]$ to 1, which suggests taking the quotient by G' . And indeed we have the following proposition:

Proposition 1. G/G' is abelian.

Proof. Let $xG', yG' \in G/G'$, since $xy = yx[x, y]$ and $[x, y] \in G'$, we have $(xG')(yG') = (xyG') = (yx[x, y]G') = (yxG')(xG')$. $\square$

So we were able to “abelianize” G . However, G' is also nice in that it has the property that it is the smallest normal subgroup of G such that G/G' is abelian. In other words, G/G' is the largest abelian quotient of G . Precisely, we have the following proposition:

Proposition 2. $H \trianglelefteq G$ and G/H abelian if and only if $G' \leq H$.

Proof. First suppose $H \trianglelefteq G$ and G/H is abelian.

So $1H = (xH)^{-1}(yH)^{-1}(xH)(yH) = (x^{-1}y^{-1}xy)H = ([x, y]H)$ which means $[x, y] \in H$ for all $x, y \in G$, so $G' \leq H$.

Now suppose $G' \leq H$, then since G/G' is abelian, every subgroup is normal and so $G/H \trianglelefteq G/G'$ and by the lattice isomorphism theorem that means $H \trianglelefteq G$. Also, by the third isomorphism theorem, we have $G/H \cong (G/G')/(H/G')$ which means G/H is abelian since it is the quotient of an abelian group (G/G') . $\square$

Additionally we have this nice universal property regarding G/G' which is that for any abelian group A and group homomorphism $\phi : G \rightarrow A$, $G' \subseteq \text{Ker}\phi$ and the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G/G' \\ & \searrow \phi & \downarrow \exists! \bar{\phi} \\ & & A \end{array}$$

Proof. It is easy enough to show $G' \subseteq \text{Ker}\phi$

since $\phi(x^{-1}y^{-1}xy) = \phi(x^{-1})\phi(y^{-1})\phi(x)\phi(y) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y)$ and since A is abelian, this equals 1_A . We define $\bar{\phi} : G/G' \rightarrow A$ by $\bar{\phi}(xG') = \phi(x)$. Now we need to show $\bar{\phi}$ exists and is unique and commutes according to the diagram above. To show that it exists, we show that it is a well defined group homomorphism. Suppose $xG' = yG'$, for $\bar{\phi}$ to be well defined, we need $\bar{\phi}(xG') = \bar{\phi}(yG')$. Since $xG' = yG'$, $xy^{-1} \in G' \subseteq \text{Ker}\phi$, so $\phi(x)\phi(y)^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1}) = 1_A$, so $\phi(x) = \phi(y)$, which by the definition of $\bar{\phi}$, means $\bar{\phi}(xG') = \bar{\phi}(yG')$. $\bar{\phi}$ is indeed a group homomorphism because $\bar{\phi}(xG'yG') = \bar{\phi}(xyG') = \phi(xy) = \phi(x)\phi(y) = \bar{\phi}(xG')\bar{\phi}(yG')$ and $\bar{\phi}$ is unique because it is completely determined and defined by ϕ . Finally, $\phi = \bar{\phi} \circ \eta$ because $\bar{\phi}(\eta(x)) = \bar{\phi}(xG') = \phi(x)$. $\square$

References

- [DF04] David Dummit and Richard Foote. *Abstract Algebra*. John Wiley & Sons, Inc., third edition, 2004.