2-Categories

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Abstract

Some introductory notes on 2-Categories.

The key thing to remember is that a 2-Category is just a category enriched in **Cat**, the category of small categories (A normal category is a category enriched in **Set**). So the hom-sets of a 2-Category are themselves a small category. First let's recast the definition of a category in terms of diagrams:

Definition (Category). A (locally small) category \mathscr{A} consists of a collection of

- 1. objects A, B, C, \ldots
- 2. sets of morphisms $\mathscr{A}(A, B), \mathscr{A}(B, C), \ldots$

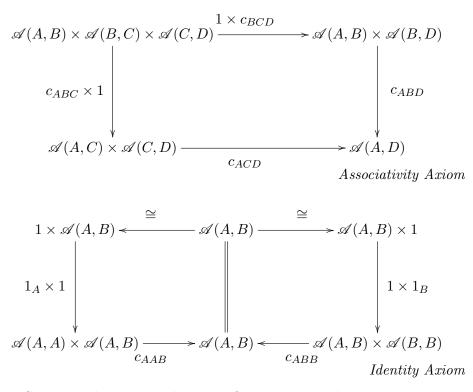
such that for every object A, there is a function, called *identity*

$$1_A: 1 \to \mathscr{A}(A, A)$$

and for every triple of objects A, B, C, there is a function, called *composition*

 $c_{ABC}: \mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{A}(A,C)$

These functions obey the following commutative diagrams:



2-Categories have an analogous definition except that since we are enriched in **Cat**, $\mathscr{A}(A, B)$ are no longer sets, but rather small categories. Equivalently, 1_A , c_{ABC} are no longer functions but rather functors. With that, we have the following definition:

Definition (2-Category). A 2-Category \mathscr{A} consists of a collection of

- 1. objects, called 0-cells A, B, C, \ldots
- 2. for each pair of 0-cells A, B a small category $\mathscr{A}(A, B)$ where the objects of $\mathscr{A}(A, B)$ are our familiar morphisms $f: A \to B$, also called 1-cells, and the morphisms in $\mathscr{A}(A, B)$, $\alpha : f \Rightarrow g$ are called 2-cells.
- 3. for each 0-cell A, a functor

$$1_A: 1 \to \mathscr{A}(A, A)$$

4. for each triple of 0-cells A, B, C a bifunctor

 $c_{ABC}: \mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{A}(A,C)$

such that the associativity and identity diagrams above commute.

Since $\mathscr{A}(A, B)$ is a category and composition is a functor now, there's more stuff going on, which we need to take a look at:

References

[Bor94] Francis Borceux. Handbook of Categorical Algebra, volume 1 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.