2-Categories

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Abstract

Some introductory notes on 2-Categories.

The key thing to remember is that a 2-Category is just a category enriched in **Cat**, the category of small categories (A normal category is a category enriched in **Set**). So the hom-sets of a 2-Category are themselves a small category. First let's recast the definition of a category in terms of diagrams:

Definition (Category)**.** A (locally small) category $\mathscr{A}$ consists of a collection of

1. objects $A, B, C, \ldots$
2. sets of morphisms $\mathscr{A}(A, B), \mathscr{A}(B, C), \ldots$

such that for every object A, there is a function, called *identity*

$$1_A : 1 \to \mathscr{A}(A, A)$$

and for every triple of objects A, B, C, there is a function, called *composition*

$$c_{ABC} : \mathscr{A}(A, B) \times \mathscr{A}(B, C) \to \mathscr{A}(A, C)$$

These functions obey the following commutative diagrams:

$$\begin{CD} \mathscr{A}(A,B)\times\mathscr{A}(B,C)\times\mathscr{A}(C,D) @>{1\times c_{BCD}}>> \mathscr{A}(A,B)\times\mathscr{A}(B,D) \\ @V{c_{ABC}\times 1}VV @VV{c_{ABD}}V \\ \mathscr{A}(A,C)\times\mathscr{A}(C,D) @>>{c_{ACD}}> \mathscr{A}(A,D) \end{CD}$$

Associativity Axiom

$$\begin{CD} 1\times\mathscr{A}(A,B) @<{\cong}<< \mathscr{A}(A,B) @>{\cong}>> \mathscr{A}(A,B)\times 1 \\ @V{1_A\times 1}VV @| @VV{1\times 1_B}V \\ \mathscr{A}(A,A)\times\mathscr{A}(A,B) @>>{c_{AAB}}> \mathscr{A}(A,B) @<<{c_{ABB}}< \mathscr{A}(A,B)\times\mathscr{A}(B,B) \end{CD}$$

Identity Axiom

2-Categories have an analogous definition except that since we are enriched in **Cat**, $\mathscr{A}(A,B)$ are no longer sets, but rather small categories. Equivalently, $1_A, c_{ABC}$ are no longer functions but rather functors. With that, we have the following definition:

Definition (2-Category)**.** A 2-Category $\mathscr{A}$ consists of a collection of

1. objects, called 0-cells $A, B, C, \ldots$
2. for each pair of 0-cells A, B a small category $\mathscr{A}(A,B)$ where the objects of $\mathscr{A}(A,B)$ are our familiar morphisms $f : A \to B$, also called 1-cells, and the morphisms in $\mathscr{A}(A,B)$, $\alpha : f \Rightarrow g$ are called 2-cells.
3. for each 0-cell A, a functor

$$1_A : 1 \to \mathscr{A}(A,A)$$

4. for each triple of 0-cells A, B, C a bifunctor

$$c_{ABC} : \mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{A}(A,C)$$

such that the associativity and identity diagrams above commute.

Since $\mathscr{A}(A, B)$ is a category and composition is a functor now, there's more stuff going on, which we need to take a look at:

References

[Bor94] Francis Borceux. *Handbook of Categorical Algebra*, volume 1 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994.